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ABSTRACT

This is part three of a three-part manual for teachers using SMSG high school text materials. Detailed solutions are given to all the exercises in the text. Chapter topics include: (1) area and the integral; and (2) differentiation theory and technique. (MP)

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CALCULUS OF ELEMENTARY FUNCTIONS

Part III

Teacher's Commentary

(Preliminary Edition)

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Chapter 7

AREA AND THE INTEGRAL

Solutions Exercises 7-1

1. $x^3(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}) < A(x) < x^3(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2})$

   $f : x \to x^2$

   (a) $(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}) < A(1) < (\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2})$

   (i) $n = 5$ $\quad \frac{36}{150} < A(1) < \frac{66}{150}$

   $.24 = \frac{6}{25} < A(1) < \frac{11}{25} = .44$

   (ii) $n = 100$ $\quad \frac{19,701}{60,000} < A(1) < \frac{20,301}{60,000}$

   $.328 \approx \frac{6,567}{20,000} < A(1) < \frac{6,767}{20,000} \approx .338$

   (b) $2^3(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}) < A(2) < 2^3(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2})$

   (i) $n = 5$ $\quad 8(\frac{6}{25}) < A(2) < 8(\frac{11}{25})$

   $1.92 = \frac{48}{25} < A(2) < \frac{88}{25} \sim 3.52$

   (ii) $n = 100$ $\quad 8(\frac{6,567}{20,000}) < A(2) < 8(\frac{6,767}{20,000})$

   $2.63 \sim \frac{6,567}{2,500} < A(2) < \frac{6,767}{2,500} \sim 2.71$

   (c) $f : x \to x^2$ and $A : x \to \frac{1}{3}x^3$

   (i) If $x = \frac{1}{2}$, then $A(\frac{1}{2}) = \frac{1}{24}$

   (ii) If $x = 3\sqrt{3}$, then $A(3\sqrt{3}) = 27\sqrt{3}$

2. (a) Sum of the areas of the interior rectangles: $[f : x \rightarrow x^3]$

$$\frac{x}{n}[f(0) + f(\frac{x}{n}) + f(\frac{2x}{n}) + \dots + f(\frac{(n-1)x}{n})]$$

$$\frac{x}{n}[0 + \frac{x^3}{n^3} + \frac{2^3x^3}{n^3} + \frac{3^3x^3}{n^3} + \dots + \frac{(n-1)^3x^3}{n^3}]$$

$$\frac{x^4}{n^4}[0 + 1^3 + 2^3 + 3^3 + \dots + (n-1)^3]$$

$$\frac{x^4}{n^4} \cdot \frac{n^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

$$\frac{x^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

Sum of $(n - 1)$ cubes:

$$(\frac{(n-1)n}{2})^2$$

$$\frac{(n-1)^2n^2}{4}$$

$$\frac{n^4}{4}(1 - \frac{2}{n} + \frac{1}{n^2})$$

(b) Sum of the areas of the exterior rectangles: $[f : x \rightarrow x^3]$

$$\frac{x}{n}[f(\frac{x}{n}) + f(\frac{2x}{n}) + \dots + f(\frac{nx}{n})]$$

$$\frac{x}{n}[\frac{x^3}{n^3} + \frac{2^3x^3}{n^3} + \dots + \frac{n^3x^3}{n^3}]$$

$$\frac{x^4}{n^4}[1^3 + 2^3 + 3^3 + \dots + n^3]$$

$$\frac{x^4}{n^4} \cdot \frac{n^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

$$\frac{x^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

Sum of n cubes:

$$(\frac{n(n+1)}{2})^2$$

$$\frac{n^4}{4}(1 + \frac{2}{n} + \frac{1}{n^2})$$

$\therefore$ Summarizing part (c) and part (b), we have

$$\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

When $n \to \infty$, we have $\frac{x^4}{4} \leq A(x) \leq \frac{x^4}{4}$; i.e., $A : x \to \frac{1}{4} x^4$.

(c) $\frac{x^4}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(x) < \frac{x^4}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

$\frac{1}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) < A(1) < \frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$

(i) $n = 5$ $\quad .16 = \frac{4}{25} < A(1) < \frac{9}{25} = .36$

(ii) $n = 100$ $\quad \frac{9,801}{40,000} < A(1) < \frac{10,201}{40,000}$

$.245 < A(1) < .255$

(d) $16\left[\frac{1}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)\right] < A(2) < 16\left[\frac{1}{4}\left(1 - \frac{2}{n} + \frac{1}{n^2}\right)\right]$

$n = 100$ $\quad \frac{9,801}{2,500} < A(2) < \frac{10,201}{2,500}$

$3.92 < A(2) < 4.08$

(e) $f : x \to x^3$ and $A : x \to \frac{1}{4} x^4$

(i) If $x = 0.4$, then $A(0.4) = 0.0064$

(ii) If $x = 5\sqrt{2}$, then $A(5\sqrt{2}) = 625$

3. The area under the curve $y = 1$: $A_0^1 = 1$

The area under the curve $y = x^3$: $A_0^1 = \frac{1}{4}$

Therefore, the area of the shaded region is $1 - \frac{1}{4} = \frac{3}{4}$.

4. The intersection points are $(0,0)$ and $(1,0)$.

The area under the curve $y = x$: $A_0^1 = \frac{1}{2}$

The area under the curve $y = x^2$: $A_0^1 = \frac{1}{3}$

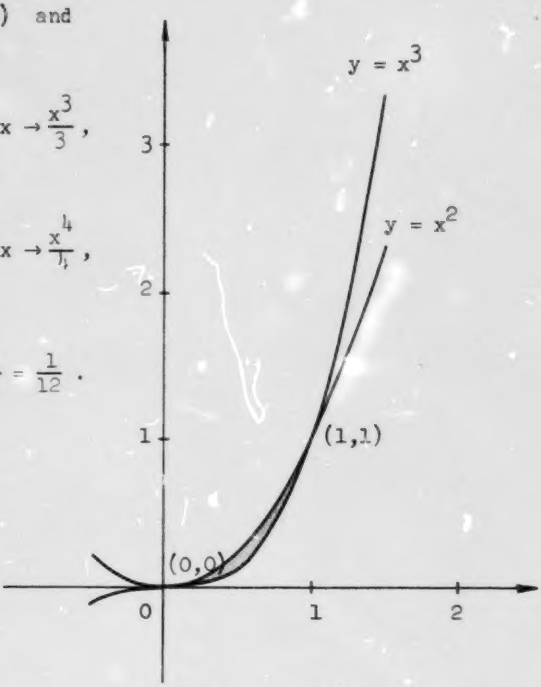
The area between the curves is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

5. Intersection points are $(0,0)$ and $(1,1)$

$$\begin{cases} \text{If } f : x \to x^2 \text{ and } A : x \to \dfrac{x^3}{3}, \\ \text{then } A(1) = \dfrac{1}{3}. \end{cases}$$

$$\begin{cases} \text{If } g : x \to x^3 \text{ and } A : x \to \dfrac{x^4}{4}, \\ \text{then } A(1) = \dfrac{1}{4}. \end{cases}$$

Area of shaded region $= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.



6. If $f : x \to ax^2 + bx + c$

$$f(0) = 0 + 0 + c$$

$$f(\tfrac{x}{n}) = a(\tfrac{x}{n})^2 + b(\tfrac{x}{n}) + c$$

$$f(\tfrac{2x}{n}) = a(\tfrac{2x}{n})^2 + b(\tfrac{2x}{n}) + c$$

$$f(\tfrac{3x}{n}) = a(\tfrac{3x}{n})^2 + b(\tfrac{3x}{n}) + c$$

. . . .

$$f(\tfrac{(n-1)x}{n}) = a(\tfrac{(n-1)x}{n})^2 + b(\tfrac{(n-1)x}{n}) + c$$

$$f(\tfrac{nx}{n}) = a(\tfrac{nx}{n})^2 + b(\tfrac{nx}{n}) + c$$

Sum of areas of interior rectangles:

$$\frac{x}{n}\{a[0 + \frac{x^2}{n^2} + \frac{2^2x^2}{n^2} + \frac{3^2x^2}{n^2} + \ldots + \frac{(n-1)^2x^2}{n^2}] + b[0 + \frac{x}{n} + \frac{2x}{n} + \ldots + \frac{(n-1)x}{n}] + cn\}$$

Sum of areas of exterior rectangles:

$$\frac{x}{n}\{a[\frac{x^2}{n^2} + \frac{2^2x^2}{n^2} + \frac{3^2x^2}{n^2} + \ldots + \frac{(n-1)^2x^2}{n^2} + \frac{n^2x^2}{n^2}] + b[\frac{x}{n} + \frac{2x}{n} + \frac{3x}{n} + \ldots + \frac{(n-1)x}{n}$$
$$+ \frac{nx}{n}] + cn\}$$

It follows that the sum of the area of the interior rectangles simplifies to

$$\frac{x}{n}\{\frac{ax^2}{n^2}[0+1^2+2^2+3^2+\ldots+(n-1)^2]+\frac{bx}{n}[0+1+2+3+\ldots+(n-1)]+[cn]\}$$

$$\frac{ax^3}{n^3}[n^3(\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2})]+\frac{bx^2}{n^2}[\frac{(n-1)n}{2}]+cn\,\frac{x}{n}$$

$$\frac{ax^3}{3}(1-\frac{3}{2n}+\frac{1}{2n})+\frac{bx^2}{2}(1-\frac{1}{n})+cx;$$

it follows, similarly, that the sum of the areas of the exterior rectangles simplifies to

$$\frac{x}{n}\{\frac{ax^2}{n^2}[1^2+2^2+3^2+\ldots+n^2]+\frac{bx}{n}[1+2+3+\ldots+n]+cn\}$$

$$\frac{ax^3}{n^3}[n^3(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2})]+\frac{bx^2}{n^2}[\frac{n(n+1)}{2}]+[cx]$$

$$\frac{ax^3}{3}(1+\frac{3}{2n}+\frac{1}{2n})+\frac{bx^2}{2}(1+\frac{1}{n})+cx.$$

Therefore, we may now write the inequality

$$\frac{ax^3}{3}(1-\frac{3}{2n}+\frac{1}{2n})+\frac{bx^2}{2}(1-\frac{1}{n})+cx<\underline{A(x)}<\frac{ax^3}{3}(1+\frac{3}{2n}+\frac{1}{2n})+\frac{bx^2}{2}(1+\frac{1}{n})+cx.$$

If $n\to\infty$, we have $\frac{ax^3}{3}+\frac{bx^2}{2}+cx\le A(x)\le\frac{ax^3}{3}+\frac{bx^2}{2}+cx$, i.e., if $f:x\to ax^2+bx+c$, then under the conditions of this problem the area in the first quadrant is

$$A:x\to\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx.$$

Note: Since we know from this section that the area function $A:x\to cx$ corresponds to the function $x\to c$, and that the area function $A:x\to\frac{bx^2}{2}$ corresponds to the function $x\to bx$, we need only work out the area function $A:x\to\frac{1}{3}ax^3$, corresponding to the function $x\to ax^2$. Assuming that the area function of a sum is the sum of area functions, then for the function,

$$f:x\to ax^2+bx+c$$

the corresponding area function is

$$A:x\to\frac{ax^3}{3}+\frac{bx^2}{2}+cx.$$

7. If $a > 0$ then $c \geq 0$.

When $a > 0$ the parabola is convex. Thus, all that is necessary is that the y-intercept be non-negative.

If $a < 0$ then $c > 0$.

When $a < 0$ the parabola is concave. Thus, the y-intercept must be greater than zero.

8. (a) $f : x \to x^2 + 1$; $a = 1 > 0$ and $c = 1 \geq 0$

There is a non-empty region in the first quadrant.

(b) $f : x \to x^2 - 2x$; $a = 1 > 0$ and $c = 0 \geq 0$

There is a non-empty region in the first quadrant.

(c) $f : x \to 2x - 3x^2$; $a = -3 < 0$ and $c = 0 \not> 0$

No region occurs in the first quadrant.

(d) $f : x \to x - 1 - x^2$; $a = -1 < 0$ and $c = -1 \not> 0$

No region occurs in the first quadrant.

9. (a) $F : x \to 2x^2 + 6x + 3$; $A : x \to \frac{2}{3}x^3 + 3x^2 + 3x$

(i) $A(1) = 6\frac{2}{3}$

(ii) $A(3) = 54$

(b) $f : x \to 12x^2 + 38x + 16$; $A : x \to 4x^3 + 19x^2 + 16x$

(i) $A(\frac{1}{2}) = 13\frac{1}{4}$

(ii) $A(1) = 39$

(c) $f : x \to -3x^2 + 18x + 12$; $A : x \to -x^3 + 9x^2 + 12x$

(i) $A(0) = 0$

(ii) $A(1) = 20$

(iii) $A(2) = 52$

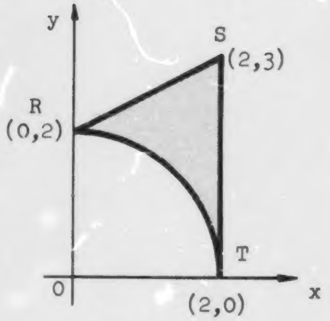
(iv) $A(4) = 128$

10. $f : x \to \frac{x}{2} + 2$; $A : x \to \frac{1}{4}x^2 + 2x$

$$A(2) = 1 + 4 = 5$$

$$\left\{\begin{array}{l}\text{Area of quarter circle}\\ \frac{\pi(2)^2}{4} = \pi\end{array}\right.$$

$$\left\{\begin{array}{l}\text{Area of trapezoid ORST:}\\ \frac{1}{2}(2)(2 + 3) = 5\end{array}\right.$$



$\therefore$ Area of shaded region: $5 - \pi \approx 1.86$ sq. units.

11. First, find area under outer parabola in quadrant 1:

$$f : x \to -x^2 + 9; \quad A : x \to -\frac{1}{3}x^3 + 9x$$

$$A(3) = -9 + 27 = 18$$

Then, find area under inner parabola in quadrant 1:

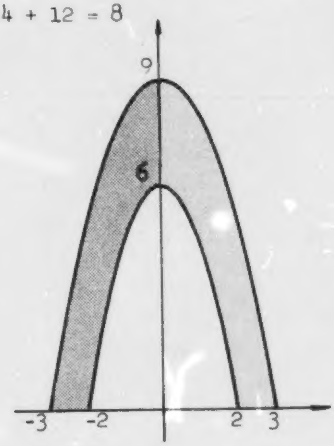
$$f : x \to -\frac{3}{2}x^2 + 6; \quad A : x \to -\frac{1}{2}x^3 + 6x$$

$$A(2) = -4 + 12 = 8$$

Therefore, by subtracting and doubling (making use of symmetry) we have:

Area of shaded region =

$2(18 - 8) = 20$ sq. units.



12. (a) We average the sum of the areas of the exterior and interior rectangles:

$$\therefore \quad A(x) \approx \frac{x^3}{3}\left(1 + \frac{1}{2n^2}\right)$$

If $n = 5$, $A(x) \approx \frac{x^3}{3}\left(1 + \frac{1}{50}\right) \approx \frac{17}{50}x^3$

(b) Adding the areas of the five trapezoids we get

$$A(x) = \frac{1}{2}\cdot\frac{x}{5}[f(0) + f(\tfrac{x}{5})] + \frac{1}{2}\cdot\frac{x}{5}[f(\tfrac{x}{5}) + f(\tfrac{2x}{5})] + \ldots + \frac{1}{2}\cdot\frac{x}{5}[f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}\cdot\frac{1}{2}[f(0) + 2f(\tfrac{x}{5}) + 2f(\tfrac{2x}{5}) + 2f(\tfrac{3x}{5}) + 2f(\tfrac{4x}{5}) + f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}[\frac{1}{2}\, f(0) + f(\tfrac{x}{5}) + f(\tfrac{2x}{5}) + f(\tfrac{3x}{5}) + f(\tfrac{4x}{5}) + \frac{1}{2}\, f(\tfrac{5x}{5})]$$

$$= \frac{x}{5}[0 + (\tfrac{x}{5})^2 + (\tfrac{2x}{5})^2 + (\tfrac{3x}{5})^2 + (\tfrac{4x}{5})^2 + \tfrac{1}{2}(\tfrac{5x}{5})^2]$$

$$= \frac{x^3}{125}(0^2 + 1^2 + 2^2 + 3^2 + 4^2 + \frac{1}{2}\, 5^2) = \frac{x^3}{125}\cdot\frac{85}{2} = \frac{17x^3}{50}$$

(c) Adding rectangles with height at midpoint of intervals:

$$A(x) = \frac{x}{5}[f(\tfrac{x}{10}) + f(\tfrac{3x}{10}) + f(\tfrac{5x}{10}) + f(\tfrac{7x}{10}) + f(\tfrac{9x}{10})]$$

$$= \frac{x}{5}[(\tfrac{x}{10})^2 + (\tfrac{3x}{10})^2 + (\tfrac{5x}{10})^2 + (\tfrac{7x}{10})^2 + (\tfrac{9x}{10})^2]$$

$$= \frac{x^3}{5\cdot 10^2}(1^2 + 3^2 + 5^2 + 7^2 + 9^2) = \frac{x^3}{5\cdot 10^2}\cdot 165 = \frac{33x^3}{100}$$

(d) Estimate (a) and (b) are the same, a fact we might suspect from elementary geometry. By comparing the fractions $\frac{17}{50}$ and $\frac{33}{100}$ to $\frac{1}{3}$, we see that the midpoint formula is slightly better than the trapezoid formula,

i.e.,

$$\frac{33}{100} < \frac{1}{3} < \frac{17}{50}$$

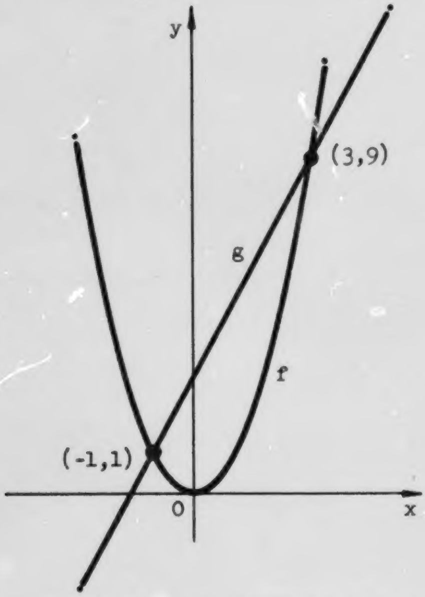
$$\frac{99}{300} < \frac{100}{300} < \frac{102}{300}\,.$$

There is an error of $\frac{1}{300}$ in using the trapezoid (or averaging interior and exterior rectangles); there is an error of $\frac{2}{300}$ in using the midpoint formula.

Solutions Exercises 7-2

1. $f : x \to x^2$, $g : x \to 2x + 3$.

(a)



(b) The graphs of f and g have two points of intersection, namely $(-1,1)$ and $(3,9)$. This implies that the graph of f is entirely above or entirely below the graph of g when $-1 < x < 3$. We pick an interior point on the interval, $x = 0$. Evaluating f and g yields $f(0) = 0 < g(0) = 3$. Thus the graph of f is entirely below the graph of g when $-1 < x < 3$ and $f(x) \leq g(x)$ when $0 \leq x \leq 3$.

(c) By (5) it is known that for $f(x) \leq g(x)$ and $a \leq x \leq b$ that $\int_a^b f \leq \int_a^b g$. We reinforce (5) by this numerical example.

For $f : x \to x^2$ then $F : x \to \frac{x^3}{3}$ is the area function.

For $g : x \to 2x + 3$ the area function is $G : x \to x^2 + 3x$.

$$\int_0^3 f = F(3) = 9$$

$$\int_0^3 g = G(3) = 18$$

Thus
$$\int_0^3 f \leq \int_0^3 g.$$

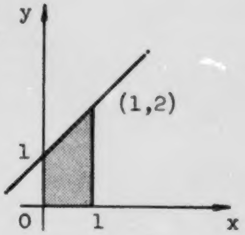
2. (a) $f : x \rightarrow x + 1$

$f': x \rightarrow 1$

We need test only the end-points since $f'(x) \neq 0$ for all values of x

$f(0) = 1 = m$

$f(1) = 2 = M$



$$m(a - 0) \leq A(1) \leq M(a - 0)$$
$$1(1 - 0) \leq A(1) \leq 2(1 - 0)$$
$$1 \leq A(1) \leq 2$$

(b) $f : x \rightarrow x^2 - 2x + 3$

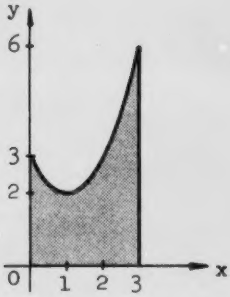
$f': x \rightarrow 2x - 2$

We must test for relative extremum, as well as end-points.

$f'(x) = 0$ when $x = 1$

$f(0) = 3$, $f(1) = 2$ and $f(3) = 6$.

Thus $m = 2$ and $M = 6$



$$2(3 - 0) \leq A(3) \leq 6(3 - 0)$$
$$6 \leq A(3) \leq 18$$

3. $f : x \rightarrow 3x - 2$ and $g = \sqrt{2}\, f$

$$\int_0^x f = \frac{3x^2}{2} - 2x$$

$$\int_5^{10} f = \int_0^{10} f - \int_0^5 f$$

$$= [\frac{3 \cdot 100}{2} - 20] - [\frac{3 \cdot 25}{2} - 10]$$

$$= \frac{205}{2}$$

$$\int_0^x g = \int_0^x \sqrt{2}\, f$$

$$= \int_0^x 3\sqrt{2}\, x - 2\sqrt{2}$$

$$= \frac{3\sqrt{2}\, x^2}{2} - 2\sqrt{2}\, x$$

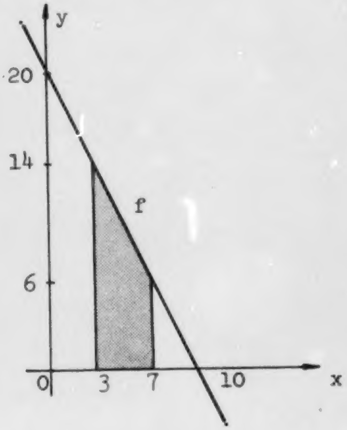
$$\int_5^{10} g = \left[\frac{3\sqrt{2}\cdot 100}{2} - 2\sqrt{2}\cdot 10\right] - \left[\frac{3\sqrt{2}\cdot 25}{2} - 2\sqrt{2}\cdot 5\right]$$

$$= \sqrt{2}\cdot\frac{205}{2}$$

Thus $$\int_5^{10} g = \sqrt{2}\int_5^{10} f.$$

4. $f : x \rightarrow -2x + 20$ and $g : x \rightarrow -2(x - h) + 20$

(a) Select $h = 3$ then $g(x) = f(x - 3)$. Substituting we find $f(3) = g(0)$, $f(7) = g(4)$.



(b) $$\int_0^x f = -x^2 + 20x$$

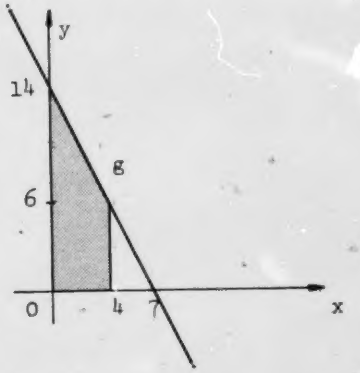
$$\int_0^x g = \int_0^x -2x - 2h + 20$$

$$= -x^2 - 2hx + 20x$$

for $h = 3$

$$\int_0^x g = -x^2 - 6x + 20x$$

$$= -x^2 + 14x$$



$$\int_0^3 f = -9 + 60$$

$$= 51$$

$$\int_0^4 g = -16 + 56$$

$$= 40$$

$$\int_0^7 f = -49 + 140$$

$$= 91$$

$$\therefore \quad \int_0^7 f = \int_0^3 f + \int_0^4 g \quad \text{and} \quad \int_0^7 f = \int_0^3 f + \int_3^7 f$$

5. $f : x \to 3x + 5$, $g : x \to x$ and $h : x \to 1$

$$\int_0^x f = \frac{3x^2}{2} + 5x$$

$$\int_0^x g = \frac{x^2}{2}$$

$$\int_0^x h = x$$

$$\int_a^b f = \int_0^b f - \int_0^a f$$

$$= (\frac{3b^2}{2} + 5b) - (\frac{3a^2}{2} + 5a)$$

$$= (\frac{3b^2}{2} - \frac{3a^2}{2}) + (5b - 5a)$$

$$= 3\int_0^b g - 3\int_0^a g + 5\int_0^b h - 5\int_0^a h$$

Thus $\int_a^b f = 3\int_a^b g + 5\int_a^b h$.

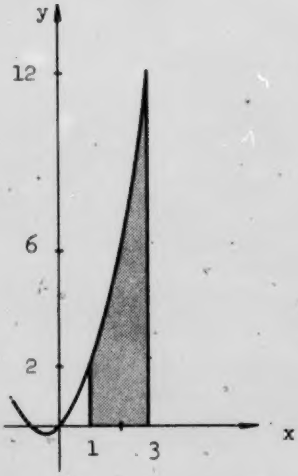
6. (a) $f : x \to x^2 + x$

$$\int_0^x f = \frac{x^3}{3} + \frac{x^2}{2}$$

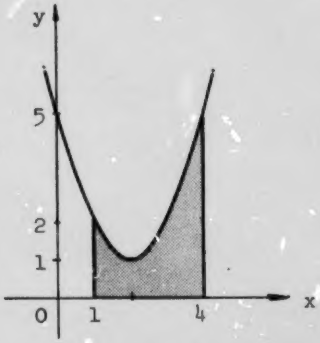
$$\int_1^3 f = \int_0^3 f - \int_0^1 f$$

$$= (9 + \frac{9}{2}) - (\frac{1}{3} + \frac{1}{2})$$

$$= \frac{38}{3}$$



(b) $f : x \to x^2 - 4x + 5$



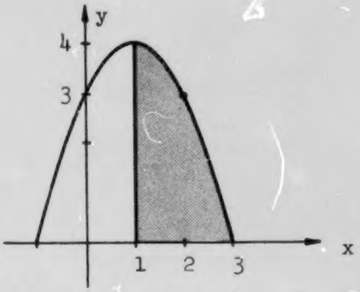
$$\int_0^x f = \frac{x^3}{3} - 2x^2 + 5x$$

$$\int_1^4 f = \int_0^4 f - \int_0^1 f$$

$$= (\frac{64}{3} - 32 + 20) - (\frac{1}{3} - 2 + 5)$$

$$= 6$$

(c) $f : x \to -x^2 + 2x + 3$



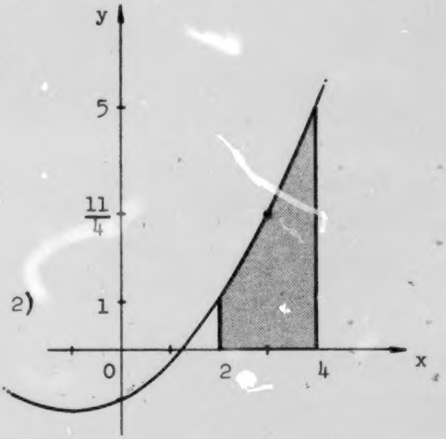
$$\int_0^x f = -\frac{x^3}{3} + x^2 + 3x$$

$$\int_1^3 f = \int_0^3 f - \int_0^1 f$$

$$= (-9 + 9 + 9) - (-\frac{1}{3} + 1 + 3)$$

$$= \frac{16}{3}$$

(d) $f : x \to \frac{1}{4}x^2 + \frac{1}{2}x - 1$



$$\int_0^x f = \frac{x^3}{12} + \frac{x^2}{4} - x$$

$$\int_2^4 f = \int_0^4 f - \int_0^2 f$$

$$= (\frac{64}{12} + \frac{16}{4} - 4) - (\frac{8}{12} + \frac{4}{4} - 2)$$

$$= \frac{17}{3}$$

7. $f : x \to px^2 + qx + r$

(a) $F : x \to \frac{p}{3}x^3 + \frac{q}{2}x^2 + rx$.

$$F'(x) = 3(\frac{p}{3})x^2 + 2(\frac{q}{2})x + r$$

$$= px^2 + qx + r$$

$$= f(x)$$

(b) $\int_a^b f = \int_0^b f - \int_0^a f$

Since $F'(x) = f(x)$.

Then $\int_0^x f = F(x)$ and $\int_a^b f = F(b) - F(a)$.

8. $g : x \to px^3 + qx^2 + rx + s$

$G : x \to \frac{p}{4}x^4 + \frac{q}{3}x^3 + \frac{r}{2}x^2 + sx$

(a) $G'(x) = 4(\frac{p}{4})x^3 + 3(\frac{q}{3})x^2 + 2(\frac{r}{2})x + s$

$= px^3 + qx^2 + rx + s$

$= g(x)$

(b) $\int_a^b g = \int_0^b g - \int_0^a g$

Since $G'(x) = g(x)$ then $\int_0^x g = G(x)$ thus $\int_a^b g = G(b) - G(a)$.

9. $G(x) = F(x) + 100$

$G'(x) = F'(x)$ implies that $G'(x) = f(x)$ also.

Since $\int_a^b f = F(b) - F(a)$ then $\int_a^b f = G(b) - G(a)$.

10. $f : x \to |x - 2|$

$$f : x \to \begin{cases} x - 2 & \text{if } x \geq 2 \\ -x + 2 & \text{if } x < 2 \end{cases}$$

Define $f = f_1$ if $x < 2$ and $f = f_2$ if $x \geq 2$.

$f_1 : x \to -x + 2$ and $\int_0^x f_1 = -\frac{x^2}{2} + 2x$.

$f_2 : x \to x - 2$ and $\int_0^x f_2 = \frac{x^2}{2} - 2x$.

$$\int_0^5 f = \int_0^2 f_1 + \int_2^5 f_2$$

$$= \int_0^2 f_1 + \int_0^5 f_2 - \int_0^2 f_2$$

$$= (-2 + 4) + (\frac{25}{2} - 10) - (2 - 4)$$

$$= \frac{13}{2}$$

11. $f : x \to x^2$

We want $g(0) = f(-10)$ and $g(7) = f(-3)$. The translation necessary is $g(x) = f(x + 10)$. Then

$$\int_{-10}^{-3} f = \int_{0}^{7} g.$$

$$g : x \to (x - 10)^2 = x^2 - 20x + 100$$

$$\int_0^x g = \frac{x^3}{3} - 10x^2 + 100x$$

$$\int_0^7 g = \frac{343}{3} - 490 + 700$$

$$= \frac{973}{3}$$

12. Let $g(x) = f(x - a)$ then

$$\int_a^b f = \int_0^{b-a} g.$$

Since $a \leq b$ then $0 \leq b - a$

$$g : x \to p(x - a)^2 + q(x - a) + r = p(x^2 - 2ax + a^2) + q(x - a) + r$$

$$= px^2 + (q - 2pa)x + pa^2 - qa + r$$

$$\int_0^x g = \frac{p}{3}x^3 + \frac{q - 2pa}{2}x^2 + (pa^2 - qa + r)x$$

$$\int_0^{b-a} g = \frac{p}{3}(b - a)^3 + \frac{q - 2pa}{2}(b - a)^2 + (pa^2 + qa + r)(b - a)$$

$$= \frac{p}{3}(b^3 - 3b^2a + 3ba^2 + a^3)$$

$$+ \frac{q - 2pa}{2}(b^2 - 2ba + a^2)$$

$$+ (pa^2 + qa + r)(b - a)$$

Expanding and collecting yields

$$\int_0^{b-a} g = (pb^3 + qb^2 + rb) - (pa^3 + qa^2 + ra)$$

$$= F(b) - F(a)$$

Thus $\int_0^b f = F(b) - F(a)$

13. $$f(x) = \begin{cases} h(x), & a \le x \le c \\ 0, & c < x \le b \end{cases}$$

$$g(x) = \begin{cases} 0, & a \le x \le c \\ h(x), & c < x \le b \end{cases}$$

$$f(x) + g(x) = \begin{cases} h(x) + 0, & a \le x \le c \\ 0 + h(x), & c < x \le b \end{cases}$$

But $h(x) + 0 = 0 + h(x) = h(x)$

$\therefore\ f(x) + g(x) = h(x)$ for $a \le x \le b$

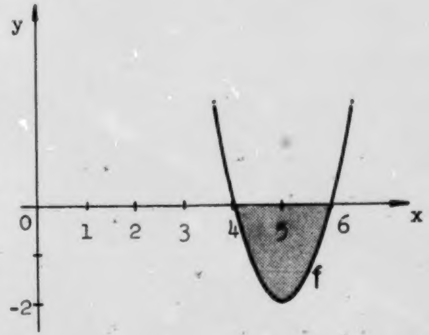
$\int_a^b (f + g) = \int_a^b f + \int_a^b g$ by (10) $\int_a^b f = \int_a^c h + \int_c^b 0 = \int_a^c h$

because $\int_c^b 0 = 0$. Similarly

$$\int_a^b g = \int_a^c 0 + \int_c^b h = \int_c^b h$$

$\therefore\ \int_a^b h = \int_a^c h + \int_c^b h.$

14. $f(x) = y = 2(x - 5)^2 - 2$ and $y = 0$.



Let $y_1 = y + 2$ in order to raise the graph two units. Then

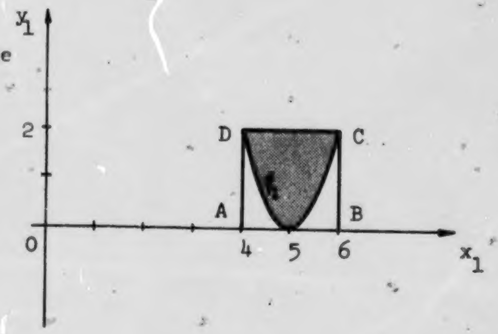
$f_1(x) = y_1 = (2(x - 5)^2 - 2) + 2$

and $f_1(x) = y_1 = 2(x - 5)^2$.

The area in question now becomes the area of $\square$ABCD less the area under the graph of y.

The area of $\square$ABCD is 4, since

$\int_0^x 2 = 2x$, $\int_4^6 2 = 12 - 8 = 4$.



The area under f_1 from 4 to 6 is $2\int_4^5 f_1$ by symmetry.

$$\int_0^x f_1 = \int_0^x 2x^2 - 20x + 50$$

$$= \frac{2}{3}x^3 - 10x^2 + 50x$$

$$\int_4^5 f_1 = (\frac{2}{3} \cdot 125 - 10 \cdot 25 + 50 \cdot 5) - (\frac{2}{3} \cdot 64 - 10 \cdot 16 + 200)$$

$$= \frac{2}{3}$$

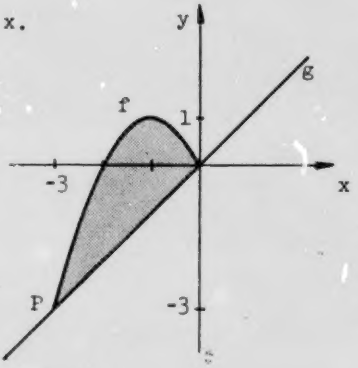
The desired area is then the area of the rectangle, 4, less

$$2\int_4^5 y = \frac{4}{3},$$

or

$$4 - \frac{4}{3} = \frac{8}{3}.$$

15. $f(x) = y = -(x + 1)^2 + 1$ and $g(x) = y = x$.



Let $y_1 = y + 3$ or $y_1 - 3 = y$ to raise the graph three units.

Let $x_1 = x + 3$ or $x_1 - 3 = x$ in order to shift the graph three units to the right.

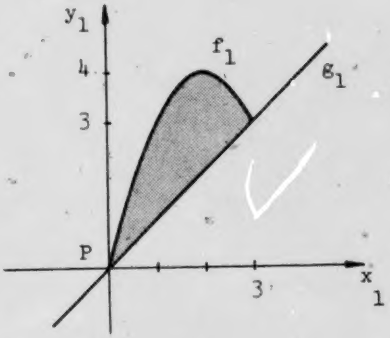
The desired area is the area found by $\int_0^3 f_1$ less the area found by $\int_0^3 g_1$

where $f_1(x_1) = y_1 = -(x - 2)^2 + 1 + 3$

$$y_1 = -(x - 2)^2 + 4$$

and $g_1(x_1) = y_1 = (x_1 - 3) + 3$

$$y_1 = x_1.$$



$$\int_0^x f_1 = \int_0^x -x_1^2 + 4x_1 - 4 + 4$$

$$= -\frac{x_1^3}{3} + 2x_1^2$$

$$\int_0^x g_1 = \int_0^x x_1$$

$$= \frac{1}{2}x_1^2$$

The desired area then is

$$\int_0^3 f_1 - \int_0^3 g_1 = (-9 + 18) - (\tfrac{9}{2})$$

$$= \tfrac{9}{2}$$

16. $A(x) = \int_a^x f$ where $a \leq x_1 \leq x_2$ and f is nonnegative.

We know that $0 \leq \int_{x_1}^{x_2} f$ for $x_1 \leq x \leq x_2$ from (4).

From (9), $$\int_a^{x_1} f + \int_{x_1}^{x_2} f = \int_a^{x_2} f.$$

Then $$\int_a^{x_1} f + \int_{x_1}^{x_2} f \leq \int_a^{x_2} f + \int_{x_1}^{x_2} f$$

or $$\int_a^{x_1} f \leq \int_a^{x_2} f$$

and $$A(x_1) \leq A(x_2).$$

Solutions Exercises 7-3

1. $A(x) \approx \frac{x^3}{3} + \frac{x^3}{6n^2}$, $n = 10$

(a) $$A(2) \approx \frac{8}{3} + \frac{8}{600}$$
$$\approx \frac{1608}{600} \approx 2.68$$

(b) $$A(2.1) \approx \frac{9.261}{3} + \frac{9.261}{600}$$
$$\approx \frac{1852.1}{600} + \frac{9.261}{600}$$
$$\approx \frac{1861}{600} \approx 3.10$$

(c) $$\frac{A(2.1) - A(2)}{0.1} \approx \frac{\frac{1861}{600} - \frac{1608}{600}}{0.1}$$
$$\approx \frac{253}{60} \approx 4.2$$

or by decimals

$$\approx \frac{3.10 - 2.68}{0.1}$$
$$\approx \frac{0.42}{0.1} = 4.2$$

(d) $$A(x + h) = \frac{(x + h)^3}{3} + \frac{(x + h)^3}{600}$$
$$= (x + h)^3\left(\frac{200 + 1}{600}\right)$$
$$= \frac{201}{600}(x + h)^3$$
$$A(x) = \frac{x^3}{3} + \frac{x^3}{600}$$
$$= \frac{201}{600}(x^3)$$
$$A(x + h) - A(x) = \frac{201}{600}[(x + h)^3 - x^3]$$
$$= \frac{201}{600}(3hx^2 + 3h^2x + h^3)$$
$$\frac{A(x + h) - A(x)}{h} = \frac{201}{600}(3x^2 + 3hx + h^2)$$

(e) $$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \frac{201}{600}(3x^2)$$
$$= \frac{201}{200} x^2$$
$$\approx x^2$$

We have found an approximation of the derivative, $A'(x) = x^2$.

2. If f is increasing and nonnegative then $0 \leq f(a) \leq f(x) \leq f(b)$ for $a \leq x \leq b$. The function f has minimum and maximum points at the end-points of the interval $[a,b]$. Minimum $f(x)$ is $f(a)$ and maximum $f(x)$ is $f(b)$. The area determined by $\int_a^b f$ is bounded by two rectangles, one too small and one too large.

$$f(a)(b - a) \leq \int_a^b f \leq f(b)(b - a).$$

3. $f : x \to x^2 + 1$ then $\int_0^x f = \frac{x^3}{3} + x$

(a) $$\lim_{h \to 0} \int_1^{1+h} f = \lim_{h \to 0} \left[\left(\frac{(1+h)^3}{3} + (1+h)\right) - \left(\frac{1}{3} + 1\right) \right]$$
$$= \lim_{h \to 0} \left[\left(\frac{1 + 3h + 3h^2 + h^3}{3} + 1 + h\right) - \left(\frac{1}{3} + 1\right) \right]$$
$$= \left(\frac{1}{3} + 1\right) - \left(\frac{1}{3} + 1\right)$$
$$= 0$$

The more alert student will see that we are integrating from one to one and immediately conclude the answer without calculations.

(b) $$\lim_{h \to 0} \frac{1}{h} \int_1^{1+h} f = \lim_{h \to 0} \left[\frac{\frac{1 + 3h + 3h^2 + h^3}{3} + (1+h) - \left(\frac{1}{3} + 1\right)}{h} \right]$$
$$= \lim_{h \to 0} \frac{6h + 3h^2 + h^2}{3h}$$
$$= \lim_{h \to 0} 2 + h + \frac{h^2}{3}$$
$$= 2$$

The observant student will see that he has taken the derivative of the area function at $x = 1$.

$$\lim_{h \to 0} \frac{1}{h} \int_1^{1+h} f = \lim_{h \to 0} \frac{A(1+h) - A(1)}{h}$$

$$= A'(1) = f(1) = 2.$$

(c) No. See comments after (a) and (b).

4. $F(x) = \int_2^x f$, where $f : x \to x^3$

$$F(x) = \frac{x^4}{4} - \frac{2^4}{4}$$

$$= \frac{x^4}{4} - 4$$

(a) $F(2) = 4 - 4$

$$= 0$$

(b) $F'(x) = 4\,\frac{x^3}{4} - 0$

$$= x^3$$

$F'(3) = 27$

(c) No. $F(2) = \int_2^2 f = 0$. No matter what function we consider, $\int_a^a f$ is always zero.

In part (b) we take the derivative of an antiderivative and evaluate at $x = 3$. By the Area Theorem $A'(x) = f(x)$.

5. By the Area Theorem, $F'(x) = f(x)$ where $F(x) = \int_a^x f$ and $x \geq a$.

(a) $F'(x) = x^4 + x^2$

(b) $F'(x) = \sin^3 x$

(c) $F'(x) = e^{\sin x}$

(d) $F'(x) = x^{100}$

6. $f : x \to \frac{1}{x}$

$$F(x) = \int_1^x f; \quad x \geq 1, \quad G(x) = \int_2^x f; \quad x \geq 2$$

The student should recall that $\int f = \log_e x$.

(a) $F(1) = 0$

$G(2) = 0$

(b) $F'(x) = G'(x) = f(x), \quad x \geq 2$

$\therefore F'(x) - G'(x) = 0$

(c) Since $F(x) = \int_1^x f$ and $G(x) = \int_2^x f$

$$F(x) = \int_1^2 f + \int_2^x f$$

$$F(x) = \alpha + G(x), \text{ where } \alpha = \int_1^2 f.$$

$\therefore F(x) - G(x) = \alpha$ for $x \geq 2$.

Since $\alpha = \log_e 2$, F and G differ by a constant.

7. (a) $f' : x \to 2x - 1$

(b) $g' : x \to 2x - 1$

(c) The derivatives are the same since the functions only differ by a constant.

8. $g' : x \to 3x^2$

$g : x \to x^3 + c$ for various values of c. The functions only differ by a constant.

9. (a) $f : x \to x^2$

$$F(x) = \int_0^x f = \frac{x^3}{3}$$

$$F(2) = \frac{8}{3}$$

(b) $f : x \to 2x + 1$

$F(x) = x^2 + x$

$F(2) = 6$

(c) $f : x \to 4x^3 + x$

$F(x) = x^4 + \frac{x^2}{2}$

$F(2) = 18$

10. $f : x \to x^2 + 1 \qquad \int_0^x f = \frac{x^3}{3} + x$

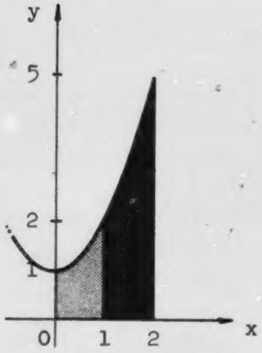
(a) See graph.

(b) $\int_0^1 f = \frac{1}{3} + 1 = \frac{4}{3}$

(c) $\int_0^2 f = \frac{8}{3} + 2 = \frac{14}{3}$

(d) The region of part (d) is equivalent to the region of part (c) with the region of part (b) removed.

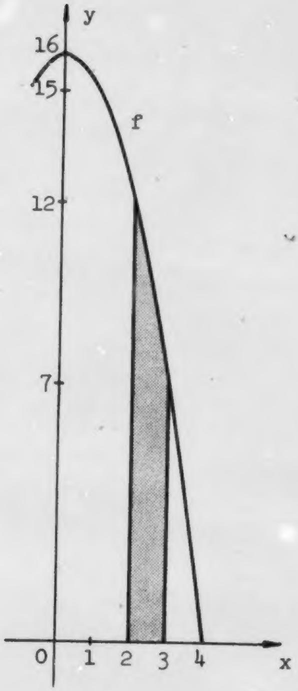
$$\int_1^2 f = \frac{4}{3} - \frac{14}{3}$$
$$= \frac{10}{3}$$



11. (a) $f : x \to 16 - x^2$

$A(x) = 16x - \frac{x^3}{3}$

$$\int_2^3 f = (48 - 9) - (32 - \frac{8}{3})$$
$$= \frac{23}{3}$$

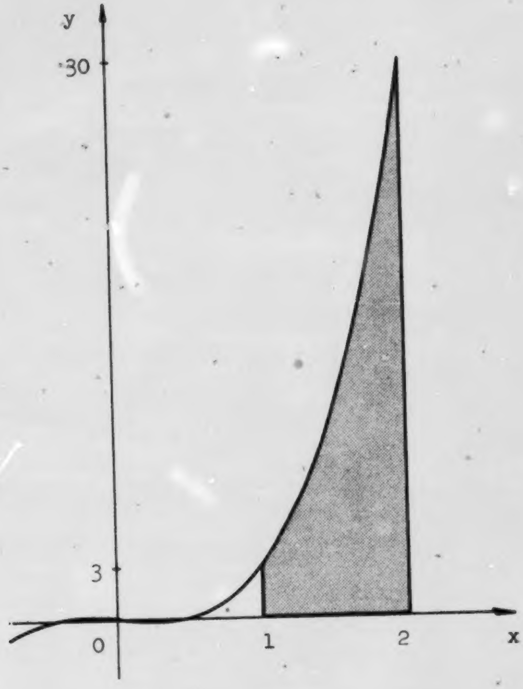


(b) $f : x \to 4x^3 - x$

$A(x) = x^4 - \frac{x^2}{2}$

$\int_1^2 f = (16 - 1) - (1 - \frac{1}{2})$

$= \frac{29}{2}$



12. (a) $f : x \to x^6$

$F : x \to \frac{x^7}{7}$

$\int_1^3 f = \frac{2187}{7} - \frac{1}{7} = \frac{2186}{7}$

(b) $f : x \to x^6 + x$

$F : x \to \frac{x^7}{7} + \frac{x^2}{2}$

$\int_1^3 f = (\frac{2187}{7} + \frac{9}{2}) - (\frac{1}{7} + \frac{1}{2})$

$= \frac{2214}{7}$

(c) $f : x \to \frac{1}{x}$

$F : x \to \log_e x$

$$\int_4^5 f = \log_e 5 - \log_e 4$$

$$= \log_e \frac{5}{4}$$

(d) $f : x \to \frac{1}{\sqrt{x}}$

$F : x \to 2\sqrt{x}$

$$\int_2^4 f = 2\sqrt{4} - 2\sqrt{2}$$

$$= 2(2 - \sqrt{2})$$

(e) $f : x \to e^x$

$F : x \to e^x$

$$\int_{-5}^0 f = e^0 - e^{-5}$$

$$= 1 - e^{-5} \quad \text{or} \quad \frac{e^5 - 1}{e^5}$$

(f) $f : x \to e^{2x}$

$F : x \to \frac{1}{2} e^{2x}$

$$\int_{-5}^0 f = \frac{1}{2} e^0 - \frac{1}{2} e^{-10}$$

$$= \frac{1}{2}(1 - e^{-10}) \quad \text{or} \quad \frac{e^{10} - 1}{2e^{10}}$$

(g) $f : x \to \sin x$

$F : x \to -\cos x$

$$\int_0^{\pi/2} f = -\cos \frac{\pi}{2} - (-\cos 0)$$

$$= 0 - (-1)$$

$$= 1$$

(h) $f : x \to \sin 2x$

$F : x \to -\frac{1}{2}\cos 2x$

$$\int_0^{\pi/2} f = -\frac{1}{2}\cos\pi - \left(-\frac{1}{2}\cos 0\right)$$

$$= -\frac{1}{2}(-1) - \left(-\frac{1}{2}(1)\right)$$

$$= 1$$

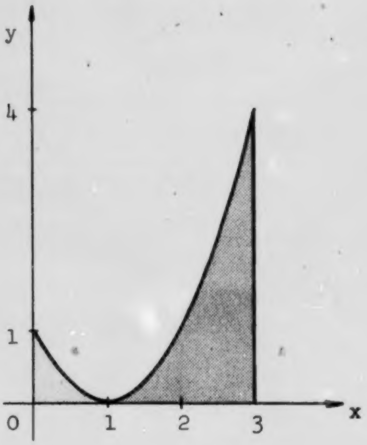
13. $f : x \to (x - 1)^2$

$$\int_0^3 f = \int_0^1 f + \int_1^3 f$$

f is decreasing when $0 \leq x < 1$

and

f is increasing when $1 < x \leq 3$.



14. $f : x \to (x - 1)^2 = x^2 - 2x + 1$

(a) $F : x \to \frac{x^2}{3} - x^2 + x$

(b) $$\int_0^1 f = F(1) - F(0)$$

$$= \left(\frac{1}{3} - 1 + 1\right) - (0)$$

$$= \frac{1}{3}$$

$$\int_1^3 f = F(3) - F(1)$$

$$= (9 - 9 + 3) - \left(\frac{1}{3}\right)$$

$$= \frac{8}{3}$$

$$\int_0^3 f = F(3) - F(0)$$
$$= (9 - 9 + 3) - (0)$$
$$= 3$$

15. (a) $f : x \to 6x^2 + 2$

$g : x \to 2x^3 + 2x$ or $2x^3 + 2x + c$ where c is some real number.

If $g : x \to 2x^3 + 2x$

$$g(2) - g(0) = 20 - 0 = 20$$

If $g : x \to 2x^3 + 2x + c$

$$g(2) - g(0) = (20 + c) - (0 - c)$$
$$= 20$$

(b) From part (a) $\int_0^2 f = g(2) - g(0)$

$$= 20$$

16. $f : x \to 4x + 3$

(a) $g : x \to 2x^2 + 3x$ or $2x^2 + 3x + c$ where c is some arbitrary real number.

If $g : x \to 2x^2 + 3x$

$$g(2) - g(1) = (8 + 6) - (2 + 3) = 9$$

If $g : x \to 2x^2 + 3x + c$

$$g(2) - g(1) = (8 + 6 + c) - (8 + 6 + c) = 9$$

(b) From part (a) $\int_1^2 f = g(2) - g(1)$

$$= 9$$

17. By the Constant Difference Theorem, since $g' = h'$ then

$$g(5) = f(5) + c$$

and $$g(2) = f(2) + c.$$

Subtracting on both sides gives

$$g(5) - g(2) = f(5) - f(2).$$

18. $F' : x \to x^3 - x^2$ and $F(0) = 0$

Then $F : x \to \frac{x^4}{4} - \frac{x^3}{3} + c$ for some c.

$$F(0) = 0$$

$$\frac{0}{4} - \frac{0}{3} + c = 0$$

Then $c = 0$ and $F : x \to \frac{x^4}{4} - \frac{x^3}{3}$.

This function is unique since there is one and only one value of c which satisfies the condition that $F(0) = 0$.

19. $G' : x \to x^3 - x^2$ and $G(0) = 1$

$G : x \to \frac{x^4}{4} - \frac{x^3}{3} + c$

$$G(0) = 1$$

$$\frac{0}{4} - \frac{0}{3} + c = 1$$

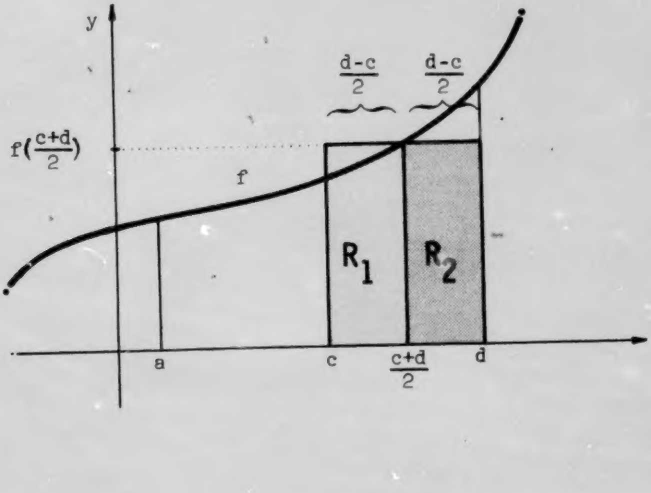
or

$$c = 1.$$

A unique G is determined namely, $G : x \to \frac{x^4}{4} - \frac{x^3}{3} + 1$.

20. f is nonnegative and increasing

$$A(x) = \int_a^x f; \quad a \leq c \leq d$$


y
$\frac{d-c}{2}$
$\frac{d-c}{2}$
$f(\frac{c+d}{2})$
f
R_1
R_2
a
c
$\frac{c+d}{2}$
d

(a) $A(\frac{c + d}{2}) = A(c) + \int_{c}^{c+d/2} f$

The area of the shaded rectangle R_1 is greater than or equal to the area under f from c to $\frac{c + d}{2}$ since the right endpoint is the maximum $f(x)$ when $c \leq x \leq \frac{c + d}{2}$.

$$\int_{c}^{c+d/2} f \leq \frac{d - c}{2} \cdot f(\frac{c + d}{2})$$

Thus $A(\frac{c + d}{2}) \leq A(c) + \frac{d - c}{2} f(\frac{c + d}{2})$.

(b) $A(d) = A(\frac{c + d}{2}) + \int_{c+d/2}^{d} f$ or $A(\frac{c + d}{2})$ [illegible] $A(d) - \int_{c+d/2}^{d} f$.

The rectangle R_2 has an area less than or equal to $\int_{c+d/2}^{d} f$ since $f(\frac{c + d}{2})$.

$$\int_{c+d/2}^{d} f \geq \frac{d - c}{2} f(\frac{c + d}{2})$$

$$-\int_{c+d/2}^{d} f \leq - \frac{d - c}{2} f(\frac{c + d}{2})$$

and $$A(\frac{c + d}{2}) \leq A(d) - \frac{d - c}{2} f(\frac{c + d}{2}).$$

(c) If the results of (a) and (b) are added together $\frac{d - c}{2} f(\frac{c + d}{2})$ vanishes leaving

$$A(\frac{c + d}{2}) \leq \frac{A(c) + A(d)}{2}.$$

Solutions Exercises 7-4

1. (a) $$\int_0^2 (x^2 + x + 3)dx = \int_0^2 x^2\,dx + \int_0^2 x\,dx + \int_0^2 3\,dx$$

$$= \frac{x^3}{3}\Big|_0^2 + \frac{x^2}{2}\Big|_0^2 + 3x\Big|_0^2$$

$$= (\frac{8}{3} - 0) + (\frac{4}{2} - 0) + (6 - 0)$$

$$= \frac{32}{3}$$

(b) $$\int_{-2}^0 (x^2 + x + 3)dx = \int_{-2}^0 x^2\,dx + \int_{-2}^0 x\,dx + \int_{-2}^0 3\,dx$$

$$= \frac{x^3}{3}\Big|_{-2}^0 + \frac{x^2}{2}\Big|_{-2}^0 + 3x\Big|_{-2}^0$$

$$= (0 - \frac{-8}{3}) + (0 - \frac{4}{2}) + (0 - (-6))$$

$$= \frac{20}{3}$$

(c) $$\int_{-2}^2 (x^2 + x + 3)dx = \int_{-2}^2 x^2\,dx + \int_{-2}^2 x\,dx + \int_{-2}^2 3dx$$

$$= \frac{x^3}{3}\Big|_{-2}^2 + \frac{x^2}{2}\Big|_{-2}^2 + 3x\Big|_{-2}^2$$

$$= (\frac{8}{3} - \frac{-8}{3}) + (\frac{4}{2} - \frac{4}{2}) + (6 - (-6))$$

$$= \frac{52}{3}$$

(d) $$\int_0^{\pi/3} \cos x\,dx = \sin x\Big|_0^{\pi/3}$$

$$= \sin\frac{\pi}{3} - \sin 0$$

$$= (\frac{\sqrt{3}}{2}) - (0)$$

$$= \frac{\sqrt{3}}{2}$$

(e) $\int_0^2 \sqrt{x^3}\,dx = \int_0^2 x^{3/2}\,dx$

$$= \frac{2}{5}x^{5/2}\Big|_0^2$$

$$= \frac{2}{5}(4\sqrt{2} - 0)$$

$$= \frac{8\sqrt{2}}{5}$$

(f) $\int_{1/16}^1 (\sqrt{x} + \sqrt[4]{x})dx = \int_{1/16}^1 x^{1/2}\,dx + \int_{1/16}^1 x^{1/4}\,dx$

$$= \frac{2}{3}x^{3/2}\Big|_{1/16}^1 + \frac{4}{5}x^{5/4}\Big|_{1/16}^1$$

$$= \frac{2}{3}(1 - \frac{1}{64}) + \frac{4}{5}(1 - \frac{1}{32})$$

$$= \frac{229}{160}$$

(g) $\int_{1/2}^1 \frac{1}{3x^2}\,dx = \int_{1/2}^1 \frac{1}{3}x^{-2}\,dx$

$$= -\frac{1}{3}x^{-1}\Big|_{1/2}^1$$

$$= (-\frac{1}{3}) - (-\frac{1}{3}\cdot\frac{2}{1})$$

$$= \frac{1}{3}$$

(h) $\int_{-2}^{-1} (5x^{-6} + x^2)dx = \int_{-2}^{-1} 5x^{-6}\,dx + \int_{-2}^{-1} x^2\,dx$

$$= -x^{-5}\Big|_{-2}^{-1} + \frac{x^3}{3}\Big|_{-2}^{-1}$$

$$= -(\frac{1}{-1} - (\frac{1}{-32})) + ((\frac{-1}{3}) - (\frac{-8}{3}))$$

$$= \frac{317}{96}$$

(i) $$\int_1^2 \frac{1}{x}\,dx = \log_e x \Big|_1^2$$

$$= \log_e 2 - \log_e 1$$

$$= \log_e 2$$

(j) $$\int_1^\pi x^\pi\,dx = \frac{x^{\pi+1}}{\pi + 1}\Big|_1^\pi$$

$$= \frac{(\pi)^{\pi+1}}{\pi + 1} - \frac{1}{\pi + 1}$$

$$= \frac{\pi^{\pi+1} - 1}{\pi + 1}$$

(k) $$\int_{-1}^2 e^x\,dx = e^x \Big|_{-1}^2$$

$$= e^2 - \frac{1}{e}$$

$$= \frac{e^3 - 1}{e}$$

(ℓ) $$\int_{-1}^2 (e^x + 1)dx = \int_{-1}^2 e^x\,dx + \int_{-1}^2 dx$$

$$= e^x \Big|_{-1}^2 + x \Big|_{-1}^2$$

$$= (e^2 - \tfrac{1}{e}) + (2 - (-1))$$

$$= \frac{e^3 - 1}{e} + 3 \quad \text{or} \quad \frac{e^3 + 3e - 1}{e}$$

(m) $$\int_{-1}^2 (e^x + x)dx = \int_{-1}^2 e^x\,dx + \int_{-1}^2 x\,dx$$

$$= e^x \Big|_{-1}^2 + \frac{x^2}{2}\Big|_{-1}^2$$

$$= (e^2 - \tfrac{1}{e}) + (2 - \tfrac{1}{2})$$

$$= \frac{2e^3 + 3e - 2}{2e}$$

(n) $$\int_1^2 (5x^4 + 3x^2 + 1)dx = \int_1^2 5x^4\, dx + \int_1^2 3x^2\, dx + \int_1^2 dx$$

$$= x^5 \Big|_1^2 + x^3 \Big|_1^2 + x \Big|_1^2$$

$$= (32 - 1) + (8 - 1) + (2 - 1)$$

$$= 39$$

(o) $$\int_{\pi/6}^{\pi/3} (\sin x + \cos x)dx = \int_{\pi/6}^{\pi/3} \sin x\, dx + \int_{\pi/6}^{\pi/3} \cos x\, dx$$

$$= -\cos x \Big|_{\pi/6}^{\pi/3} + \sin x \Big|_{\pi/6}^{\pi/3}$$

$$= ((-\tfrac{1}{2}) - (-\tfrac{\sqrt{3}}{2})) + ((\tfrac{\sqrt{3}}{2}) - (\tfrac{1}{2}))$$

$$= \sqrt{3} - 1$$

(p) $$\int_0^{4\pi/3} (e^x + \sin x)dx = \int_0^{4\pi/3} e^x\, dx + \int_0^{4\pi/3} \sin x\, dx$$

$$= e^x \Big|_0^{4\pi/3} + (-\cos x) \Big|_0^{4\pi/3}$$

$$= (e^{4\pi/3} - 1) + (-(-\tfrac{1}{2}) - (-1))$$

$$= e^{4\pi/3} + \tfrac{1}{2}$$

(q) $$\int_3^3 (x^2 + 2x + 5)dx = 0$$

(r) $$\int_{10}^{10} x^3 e^x \arctan(\sin^2 x)dx = 0$$

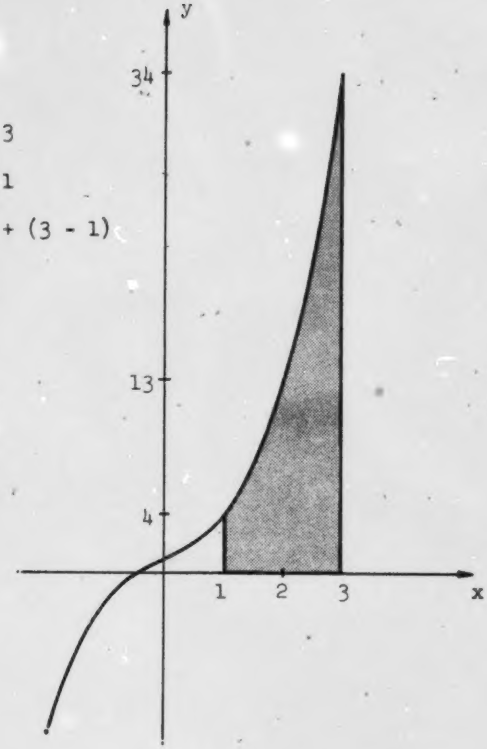
2. (a) $f : x \to x^3 + 2x + 1$

$a = 1, \quad b = 3$

$$\int_1^3 f = \frac{x^4}{4}\Bigg|_1^3 + x^2\Bigg|_1^3 + x\Bigg|_1^3$$

$$= \left(\frac{81}{4} - \frac{1}{4}\right) + (9 - 1) + (3 - 1)$$

$$= 30$$



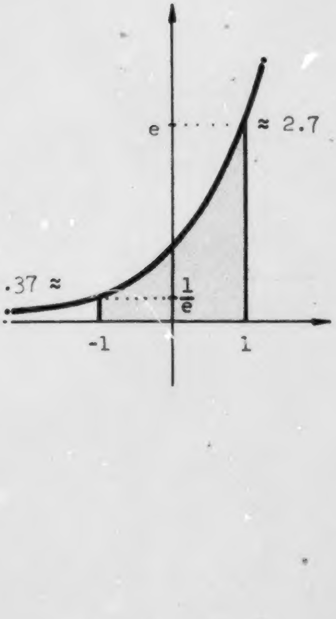
(b) $f : x \to e^x$

$a = -1, \quad b = 1$

$$\int_{-1}^1 f = e^x\Bigg|_{-1}^1$$

$$= e - \frac{1}{e}$$

$$= \frac{e^2 - 1}{e}$$



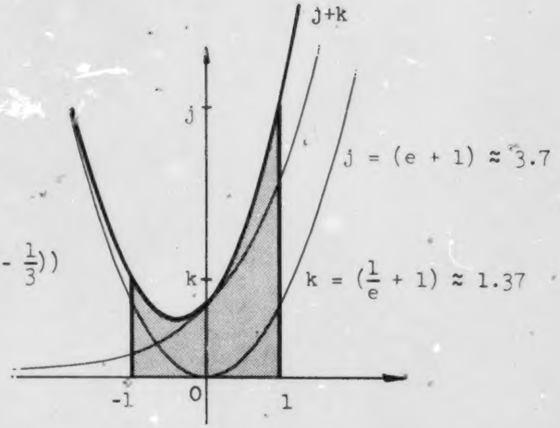
(c) $f : x \to e^x + x^2$

$a = -1, \quad b = 1$

$$\int_{-1}^{1} f = e^x \Big|_{-1}^{1} + \frac{x^3}{3} \Big|_{-1}^{1}$$

$$= \left(e - \frac{1}{e}\right) + \left(\frac{1}{3} - \left(-\frac{1}{3}\right)\right)$$

$$= \frac{3e^2 + 2e - 1}{3e}$$



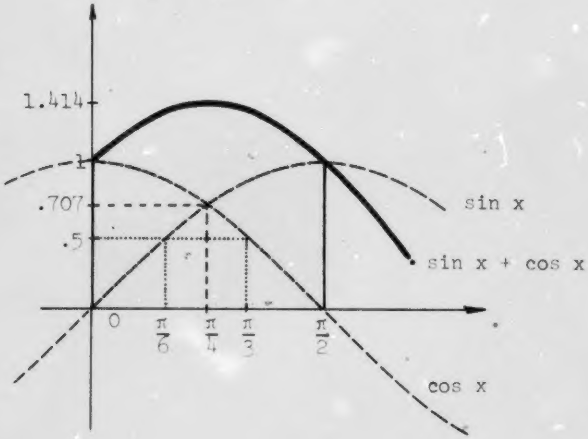
(d) $f : x \to \sin x + \cos x$

$a = 0, \quad b = \frac{\pi}{2}$

$$\int_{0}^{\pi/2} f = -\cos x \Big|_{0}^{\pi/2} + \sin x \Big|_{0}^{\pi/2}$$

$$= (-0 - (-1)) + (1 - (0))$$

$$= +2$$



(e) $f : x \to 2x^4 + \cos x$

$a = -\frac{\pi}{2}$, $b = \frac{\pi}{4}$

$$\int_{-\pi/2}^{\pi/4} f = \left(\frac{2}{5} x^5 + \sin x\right) \Big|_{-\pi/2}^{\pi/4}$$

$$= \left(\frac{2}{5}\left(\frac{\pi^5}{1024}\right) + \frac{\sqrt{2}}{2}\right) - \left(\frac{2}{5}\left(\frac{-\pi^5}{32}\right) - (-1)\right)$$

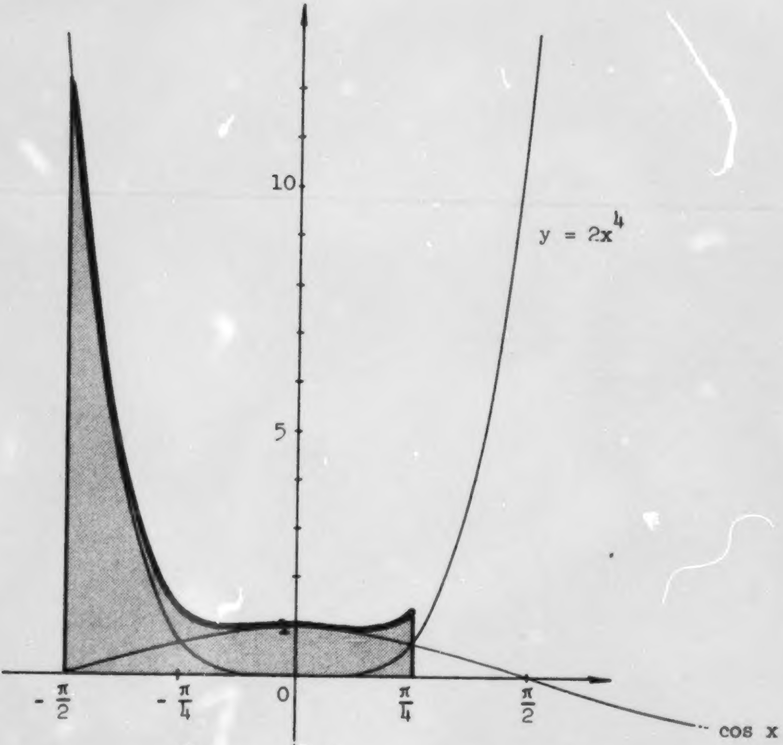
$$= \frac{2}{5}\,\frac{33}{1024}\,\pi^5 + \frac{\sqrt{2} + 2}{2}$$

$$= \frac{33\pi^5}{2560} + \frac{\sqrt{2} + 2}{2}$$

$$\approx \frac{33(305)}{2560} + \frac{3.414}{2}$$

$$\approx 3.9 + 1.707$$

$$\approx 5.6$$



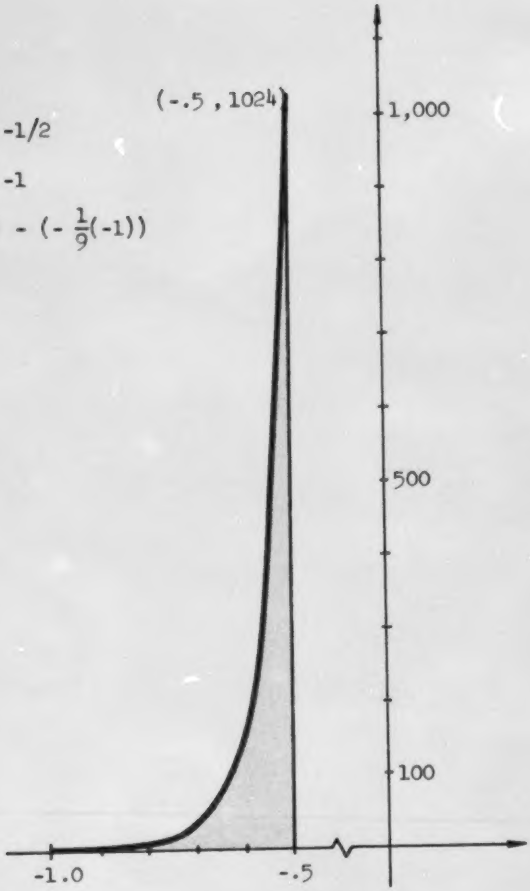
(f) $f : x \to x^{-10}$

$a = -1, \quad b = -\frac{1}{2}$

$$\int_{-1}^{-1/2} f = -\frac{1}{9}x^{-9}\Big|_{-1}^{-1/2}$$

$$= -\frac{1}{9}(-512) - \left(-\frac{1}{9}(-1)\right)$$

$$= \frac{511}{9}$$



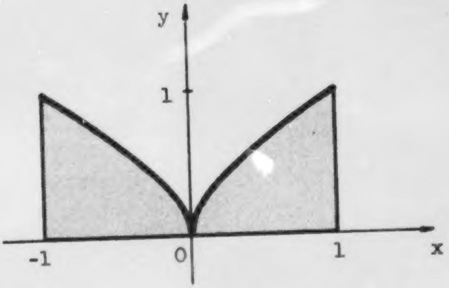
(g) $f : x \to \sqrt[3]{x^2} = x^{2/3}$

$a = -1, \quad b = 1$

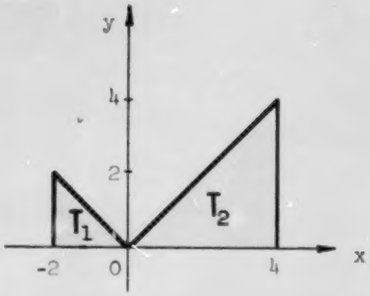
$$\int_{-1}^{1} f = \frac{3}{5}x^{5/3}\Big|_{-1}^{1}$$

$$= \frac{3}{5} - \left(-\frac{3}{5}\right)$$

$$= \frac{6}{5}$$



3. (a) $f : x \to |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



$x = -2$

$x = 4$

$$\int_{-2}^{4} f = \int_{-2}^{0} -x\,dx + \int_{0}^{4} x\,dx$$

$$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{4}$$

$$= (0 - (-2)) + (8 - 0)$$

$$= 10$$

By elementary geometry the area is equal to the area of two triangles T_1 and T_2.

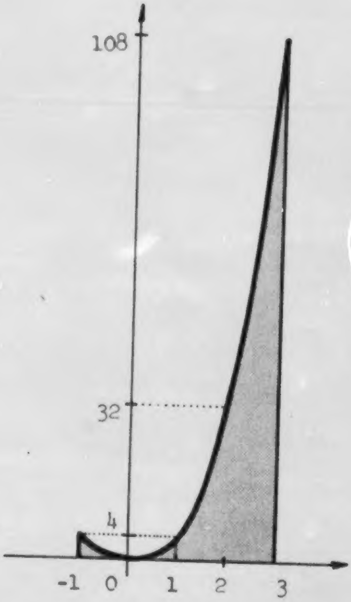
$$\text{Area} = aT_1 + aT_2$$

$$= \frac{1}{2}(2 \cdot 2) + \frac{1}{2}(4 \cdot 4)$$

$$= 2 + 8$$

$$= 10$$

(b) $f : x \to |4x^3| = \begin{cases} 4x^3, & x \geq 0 \\ -4x^3, & x < 0 \end{cases}$



$x = -1$

$x = 3$

$$\int_{-1}^{3} f = \int_{-1}^{3} -4x^3\,dx + \int_{-1}^{3} 4x^3\,dx$$

$$= -x^4\Big|_{-1}^{0} + x^4\Big|_{0}^{3}$$

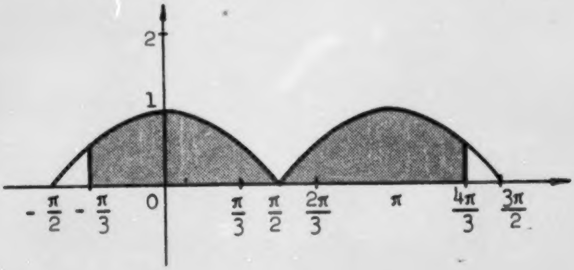
$$= (0 - (-1)) + (81 - 0)$$

$$= 82$$

(c) $f : x \rightarrow |\cos x| = \begin{cases} \cos x, & -\frac{\pi}{3} \leq x \leq \frac{\pi}{3} \\ -\cos x, & \frac{\pi}{2} \leq x \leq \frac{4\pi}{3} \end{cases}$

$x = -\frac{\pi}{3}$

$x = \frac{4\pi}{3}$



$$\int_{-\pi/3}^{4\pi/3} f = \int_{-\pi/3}^{\pi/2} \cos x \, dx + \int_{\pi/2}^{4\pi/3} - \cos x \, dx$$

$$= \sin x \Big|_{-\pi/3}^{\pi/2} + (-\sin x) \Big|_{\pi/2}^{4\pi/3}$$

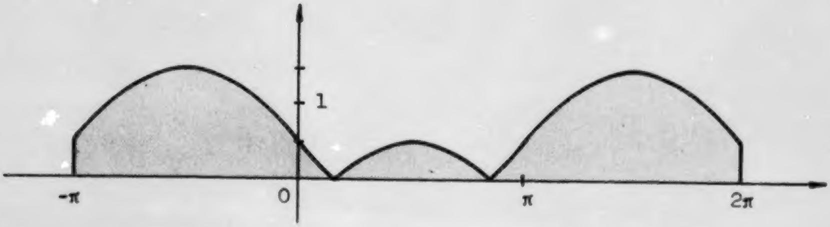
$$= (1 - (-\frac{\sqrt{3}}{2})) + ((-(-\frac{\sqrt{3}}{2}) - (-(+1)))$$

$$= 2 + \sqrt{3}$$

*(d) $f : x \rightarrow |\frac{1}{2} - \sin x| = \begin{cases} (\frac{1}{2} - \sin x), & -\pi \leq x < \frac{\pi}{6} \text{ and } \frac{5\pi}{6} \leq x \leq 2\pi \\ (-\frac{1}{2} + \sin x), & -\pi \leq x \leq \frac{\pi}{6} \end{cases}$

$x = -\pi$

$x = 2\pi$



$$\int_{-\pi}^{2\pi} f = \int_{-\pi}^{\pi/6}\left(\frac{1}{2} - \sin x\right)dx + \int_{\pi/6}^{5\pi/6}\left(-\frac{1}{2} + \sin x\right)dx + \int_{5\pi/6}^{2\pi}\left(\frac{1}{2} - \sin x\right)dx$$

$$= \left(\frac{x}{2} + \cos x\right)\Big|_{-\pi}^{\pi/6} + \left(-\frac{x}{2} - \cos x\right)\Big|_{\pi/6}^{5\pi/6} + \left(\frac{x}{2} + \cos x\right)\Big|_{5\pi/6}^{2\pi}$$

$$= \left[\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{2} + (-1)\right)\right] + \left[\left(\frac{-5\pi}{12} - \left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{-\pi}{12} - \frac{\sqrt{3}}{2}\right)\right]$$

$$+ \left[(\pi + 1) - \left(\frac{5\pi}{12} + \left(\frac{-\sqrt{3}}{2}\right)\right)\right]$$

$$= \frac{5\pi}{6} + 2 + 2\sqrt{3}$$

(e) $f : x \rightarrow |1 - \sqrt{x}| = \begin{cases} 1 - \sqrt{x}, & 0 \leq x \leq 1 \\ -1 + \sqrt{x}, & 1 < x \end{cases}$

$x = 0$

$x = 4$

$$\int_0^4 f = \int_0^1 (1 - \sqrt{x})dx + \int_1^4 (-1 + \sqrt{x})dx$$

$$= \left(x - \frac{2}{3}x^{3/2}\right)\Big|_0^1 + \left(-x + \frac{2}{3}x^{3/2}\right)\Big|_1^4$$

$$= \left[\left(1 - \frac{2}{3}\right) - 0\right] + \left[\left(-4 + \frac{16}{3}\right) - \left(-1 + \frac{2}{3}\right)\right]$$

$$= 2$$

4. (a) $(x^2 + 3\sqrt{x})\Big|_1^4 = (16 + 6) - (1 + 3)$

$= 18$

$(x^2 + 3\sqrt{x} + 50)\Big|_1^4 = (16 + 6 + 50) - (1 + 3 + 50)$

$= 18$

(b) Since F and G differ only by a constant,

$$F(x)\Big|_0^1 = (-1) - (+1) = -2 = G(x)\Big|_0^1$$

(c) If $F' = G'$ then $F(x)\Big|_a^b - G(x)\Big|_a^b = 0.$

5. (a) (i) $f : x \to (x - 1)^3$

$$\int f = \frac{1}{4}(x - 1)^4$$

(ii) $F : x \to x^3 - 3x^2 + 3x - 1$

$$\int F = \frac{1}{4} x^4 - x^3 + \frac{3}{2} x^2 - x$$

(iii) $g : x \to 8x^3 - 12x^2 + 6x - 1$

$$\int g = 2x^4 - 4x^3 + 3x^2 - x$$

(iv) $G : x \to (2x - 1)^3 = 8(x - \frac{1}{2})^3$

$$\int G = 8 \cdot \frac{1}{4}(x - \frac{1}{2})^4$$

$$= 2(x - \frac{1}{2})^4$$

(b) Since $f = F$ and $g = G$, we would expect their respective antiderivatives to differ by at most a constant.

Expanding, $$\int f = \frac{1}{4}(x^4 - 4x^3 + 6x^2 - 4x + 1)$$

$$= \frac{1}{4} x^4 - x^3 + \frac{3}{2} x^2 - x + \frac{1}{4} .$$

We note that $\int F$ and $\int f$ differ by $\frac{1}{4}$.

Expanding, $$\int G = 2(x^4 - \frac{4}{2} x^3 + \frac{6}{4} x^2 - \frac{4}{8} x + \frac{1}{16})$$

$$= 2x^4 - 4x^3 + 3x^2 - x + \frac{1}{8}$$

We see that $\int g($[illegible] and $\int G(x)$ differ by $\frac{1}{8}$.

6. $f : x \to 8(x + 1)^3$

$$\int f = 8 \frac{1}{4}(x + 1)^4$$

$$= 2(x + 1)^4$$

Since $g : x \to (2x + 2)^3 = 8(x + 1)^3$

$\int g = 2(x + 1)^4$ also.

7. Find $\int_0^1 (3x + 4)^5 \, dx$

(a) By expanding we have

$$F(x) = \int (243x^5 + 1620x^4 + 4320x^3 + 5760x^2 + 3840x + 1024)dx.$$

This is obviously a messy process which breeds arithmetic errors.

$$F(x) = \frac{243}{6} x^6 + \frac{1620}{5} x^5 + \frac{4320}{4} x^4 + \frac{5760}{3} x^3 + \frac{3840}{2} x^2 + 1024x$$

$$F(1) - F(0) = \frac{243}{6} + \frac{1620}{5} + \frac{4320}{4} + \frac{5760}{3} + \frac{3840}{2} + 1024$$

$$= \frac{243}{6} + 324 + 1080 + 1920 + 1920 + 1024$$

= 6[illegible] $\frac{1}{2}$

(b) This method should be a welcomed relief after (a).

Let $(3x + 4)^5 = 243(x + \frac{4}{3})^5$

Then $F(x) = \int 243(x + \frac{4}{3})^5 \, dx$

$$= 243 \cdot \frac{1}{6}(x + \frac{4}{3})^6$$

$$F(1) - F(0) = \frac{243}{6}(\frac{7}{3})^6 - \frac{243}{6}(\frac{4}{3})^6$$

$$= \frac{243}{6 \cdot 3^6}(7^6 - 4^6)$$

$$= \frac{1}{18}(117649 - 4096)$$

$$= \frac{1}{18}(113553)$$

$$= 6308 \frac{1}{2}$$

8. (a), (c), and (d).

9. (a) $\int_{-\pi/6}^{\pi/6} \cos x \, dx = 2\int_0^{\pi/6} \cos x \, dx$

$$= 2 \sin x \Big|_0^{\pi/6}$$

$$= 2(\frac{1}{2} - 0)$$

$$= 1$$

(b) $\int_{-2}^{2}(1 + 6x^2)dx = 2\int_{0}^{2}(1 + 6x^2)dx$

$$= 2(x + 2x^3)\Big|_0^2$$

$$= 2((2 + 16) - 0)$$

$$= 36$$

(c) $\int_{0}^{2}(x - 1)^2dx = 2\int_{0}^{1}(x - 1)^2\,dx$

$$= \frac{2}{3}(x - 1)^3\Big|_0^1$$

$$= \frac{2}{3}(0 - (-1))$$

$$= \frac{2}{3}$$

An alternate method involves changing the function so that it is symmetric with respect to the y-axis.

$$\int_{0}^{2}(x - 1)^2\,dx = \int_{-1}^{1}x^2\,dx$$

$$= 2\int_{0}^{1}x^2\,dx$$

$$= 2\,\frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3}(1 - 0)$$

$$= \frac{2}{3}$$

(d) $\int_{0}^{\pi}\sin x\,dx = 2\int_{0}^{\pi/2}\sin x\,dx$

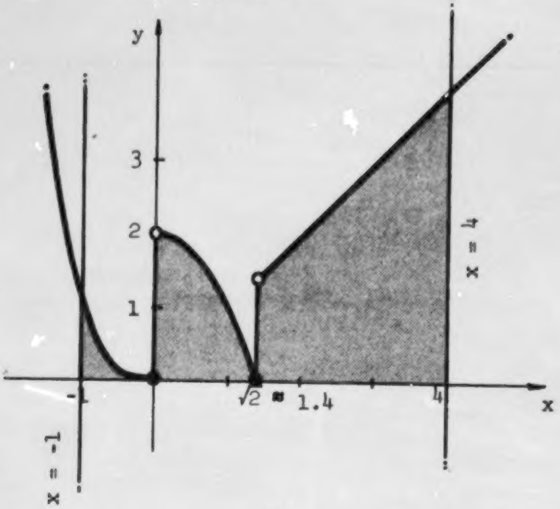
$$= -2\cos x\Big|_0^{\pi/2}$$

$$= 2(0 - (-1))$$

$$= 2$$

10. (a) $f : x \to \begin{cases} -x^3, & x \leq 0 \\ -x^2 + 2, & 0 < x \leq \sqrt{2} \\ x, & \sqrt{2} < x \end{cases}$

vertical lines $\begin{cases} x = -1 \\ x = 4 \end{cases}$



$$\int_{-1}^{4} f(x)dx = \int_{-1}^{0} -x^3 \, dx + \int_{0}^{\sqrt{2}} (-x^2 + 2)dx + \int_{\sqrt{2}}^{4} x \, dx$$

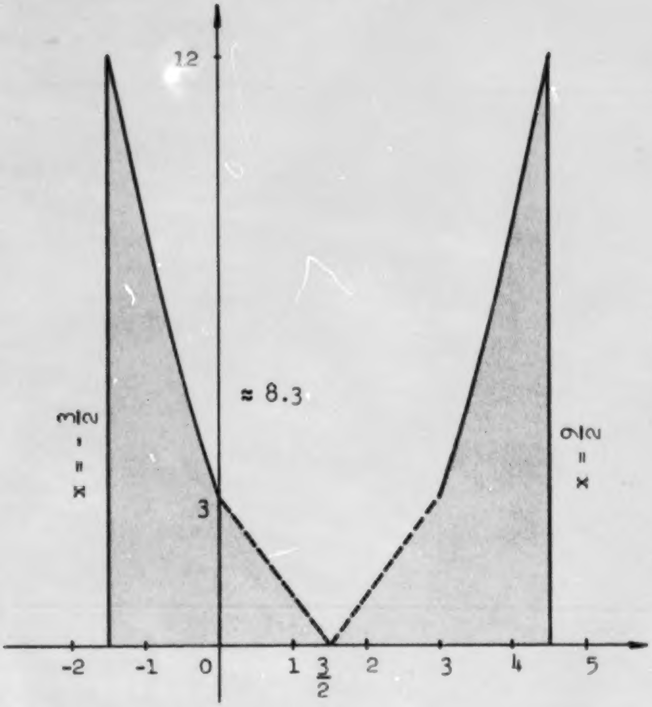
$$= -\frac{x^4}{4}\bigg|_{-1}^{0} + \left(-\frac{x^3}{3} + 2x\right)\bigg|_{0}^{\sqrt{2}} + \frac{x^2}{2}\bigg|_{\sqrt{2}}^{4}$$

$$= \left(0 - \left(-\frac{1}{4}\right)\right) + \left(\left(-\frac{2\sqrt{2}}{3} + 2\sqrt{2}\right) - 0\right) + \left(\frac{16}{2} - \frac{2}{2}\right)$$

$$= \frac{29}{4} + \frac{4}{3}\sqrt{2}$$

(b) $f : x \to \begin{cases} |2x - 3| & \text{if } 0 \leq x \leq 3 \\ \frac{4}{3}(x - \frac{3}{2})^2 & \text{if } x \leq 0 \text{ or } 3 \leq x \end{cases}$

vertical lines $\begin{cases} x = -\frac{3}{2} \\ x = \frac{9}{2} \end{cases}$



By symmetry

$$\int_{-3/2}^{9/2} f(x)dx = 2\int_{3/2}^{9/2} f(x)dx$$

$$= 2\int_{3/2}^{3} |2x - 3|dx + 2\int_{3}^{9/2} \frac{4}{3}(x - \frac{3}{2})^2 dx$$

Since $2x - 3 \geq 0$ when $x \geq \frac{2}{3}$ then

$$|2x - 3| = 2x - 3$$

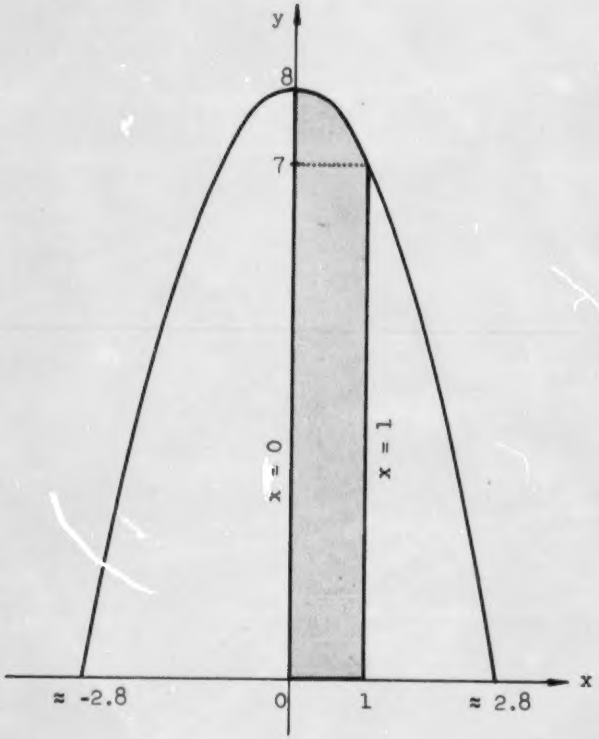
$$\int_{-3/2}^{9/2} f(x)dx = 2(x^2 - 3x) \Big|_{3/2}^{3} + \frac{8}{3}(\frac{1}{3})(x - \frac{3}{2})^3 \Big|_{3}^{9/2}$$

$$= 2((9 - 9) - (\frac{9}{4} - \frac{9}{2})) + \frac{8}{\text{[illegible]}}((\frac{6}{2})^3 - (\frac{3}{2})^3)$$

$$= \frac{51}{2}$$

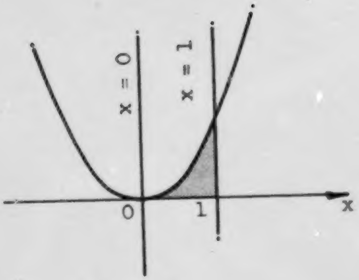
11. (a) (i) $\int_{0}^{1} (8 - x^2)dx = (8x - \frac{x^3}{3}) \Big|_{0}^{1}$

$$= \frac{23}{3}$$



(ii) $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{3}$

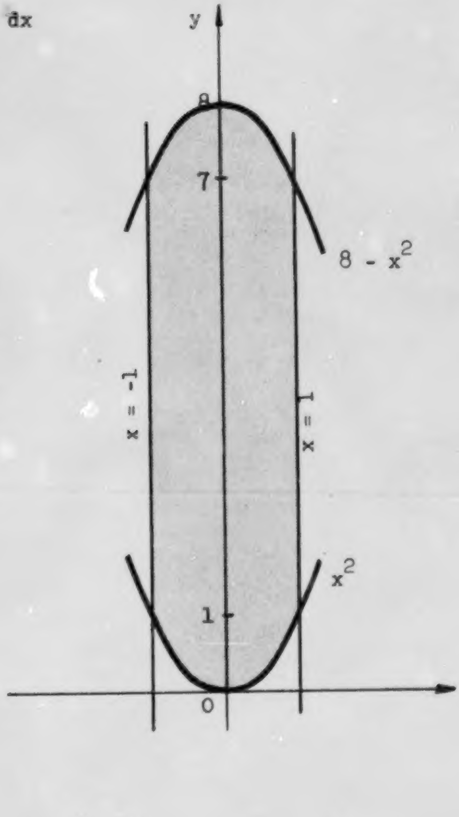


(b) The area desired is

$$2\int_0^1 (8 - x^2)dx - 2\int_0^1 x^2\,dx$$

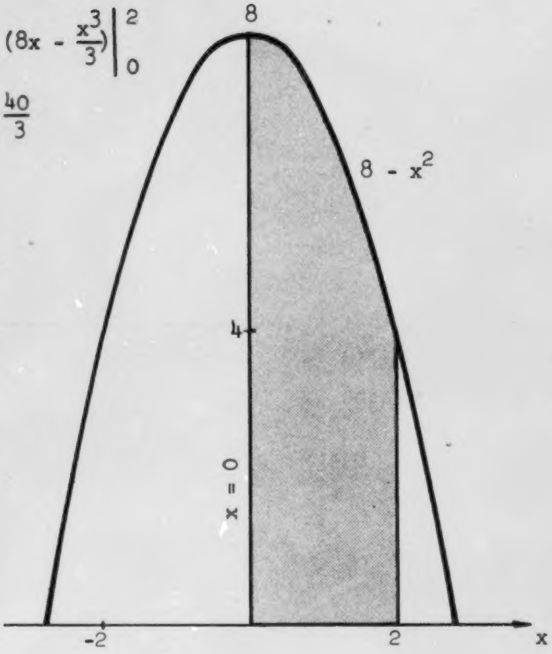
which is

$$2(\tfrac{23}{3}) - 2(\tfrac{1}{3}) = \tfrac{44}{3}$$


y
8
7
8 - x^2
x = -1
x = 1
x^2
1
0

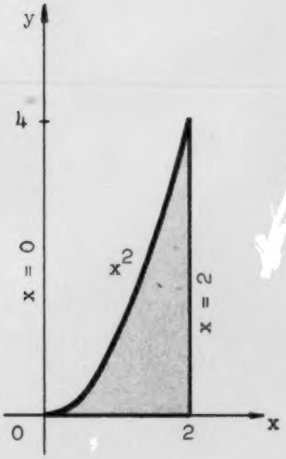
12. (a) (i) $\int_0^2 (8 - x^2)dx = (8x - \frac{x^3}{3})\Big|_0^2$

$= \frac{40}{3}$



(ii) $\int_0^2 x^2\, dx = \frac{x^3}{3}\Big|_0^2$

$= \frac{8}{3}$



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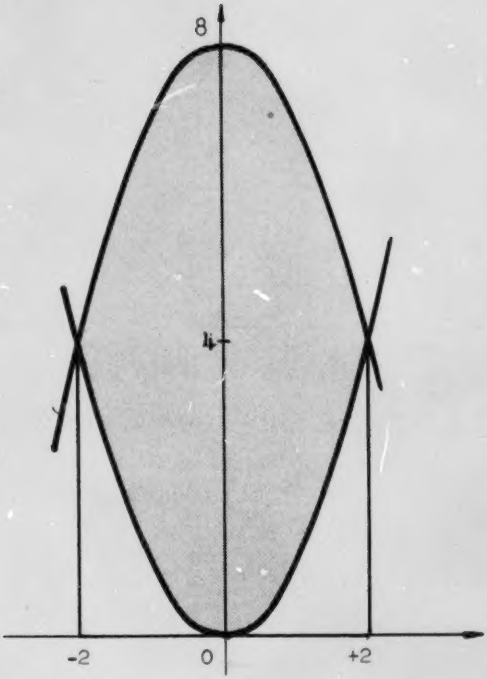
(b) The desired area is

$$2\int (8 - x^2)dx - 2\int_0^2 x^2\, dx$$

$$2(\frac{40}{3}) - 2(\frac{8}{3}) = \frac{64}{3}$$



13. (a) $$\int_{-1}^{1}(8 - x^2)dx - \int_{-1}^{1} x^2\, dx = \int_{-1}^{1}(8 - x^2 - x^2)dx$$

$$= 2\int_0^1 (8 - 2x^2)dx$$

$$= 2(8x - \frac{2}{3}x^3)\Big|_0^1$$

$$= \frac{44}{3}$$

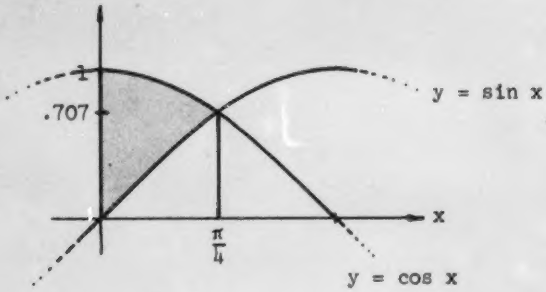
(b) $$\int_{-2}^{2}(8 - x^2)dx - \int_{-2}^{2} x^2\, dx = \int_{-2}^{2}(8 - x^2 - x^2)dx$$

$$= 2\int_0^2 (8 - 2x^2)dx$$

$$= 2(8x - \frac{2}{3}x^3)\Big|_0^2$$

$$= \frac{64}{3}$$

14. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \frac{\pi}{4}$



$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)dx$$

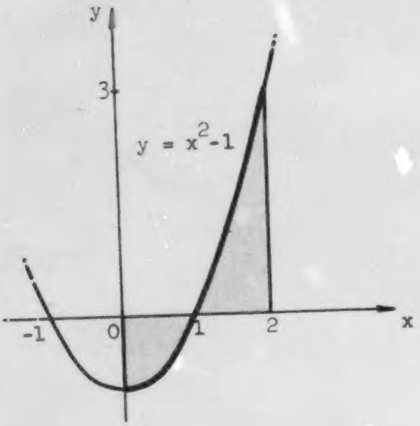
$$= (\sin x + \cos x) \Big|_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$= \sqrt{2} - 1$$

Solutions Exercises 7-5

1. (a)



(b) $\int_0^2 (x^2 - 1)dx = (\frac{x^3}{3} - x)\Big|_0^2 = \frac{8}{3} - 2 = \frac{2}{3}$

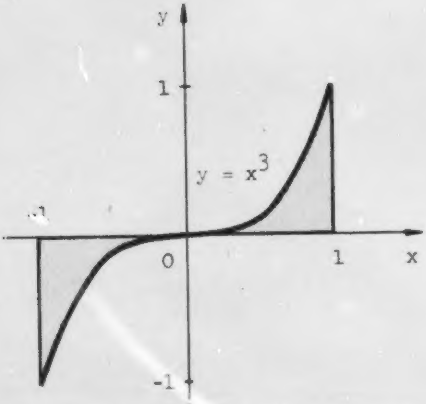
(c) $A = \int_0^1 -(x^2 - 1)dx + \int_1^2 (x^2 - 1)dx$

$= (-\frac{x^3}{3} + x)\Big|_0^1 + (\frac{x^3}{3} - x)\Big|_1^2$

$= (-\frac{1}{3} + 1) + (\frac{8}{3} - 2 - \frac{1}{3} + 1)$

$= \frac{2}{3} + \frac{4}{3} = 2$

2. (a)



(b) $\int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4 \Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

(c) $A = \int_{-1}^{0} (-x^3)dx + \int_{0}^{1} x^3\,dx$

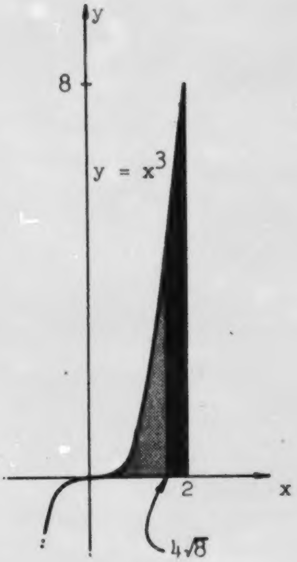
$= -\frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^4}{4}\Big|_{0}^{1}$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(d) $A = \int_{0}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{0}^{2} = 4$

$\int_{0}^{b} x^3\,dx = \frac{1}{2}\int_{0}^{2} x^3 = \frac{1}{2}(4) = 2$

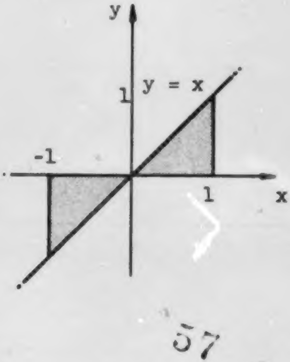
$\therefore\ \frac{1}{4}b^4 = 2$ and $b = \sqrt[4]{8} \approx 1.68$



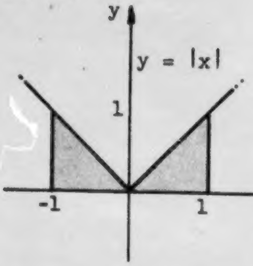
3. (a) $\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$

(b) $\int_{-1}^{1} |x|dx = \int_{-1}^{0} (-x)dx + \int_{0}^{1} x\,dx = -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$

(c) Same as (b)



(d) Same as (b) and (c).



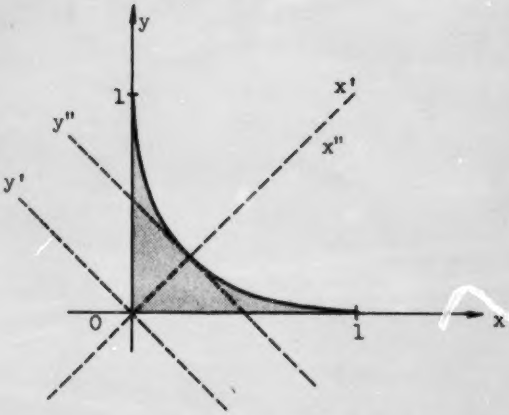
4. $\sqrt{x} + \sqrt{y} = 1$

$\therefore \quad y = 1 - 2\sqrt{x} + x$

$$A = \int_0^1 (1 - 2\sqrt{x} + x)dx$$

$$= x - \frac{4}{3}x^{3/2} + \frac{x^2}{2}\Bigg|_0^1$$

$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$



If the class has done translations and rotations, this is an excellent place to use them. Otherwise, disregard the following statements.

This is the equation of a parabola for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This can be seen by first simplifying $\sqrt{x} + \sqrt{y} = 1$:

$$x + 2\sqrt{xy} + y = 1$$

$$4xy = 1 + x^2 + y^2 - 2x - 2y + 2xy$$

$$x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$$

Since $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$, the graph is a parabola. We can rotate the axis 45° by substituting

$$x = x' \cos\theta - y' \sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = x' \sin\theta + y' \cos\theta = \frac{1}{\sqrt{2}}(x' - y')$$

in the equation $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$

$$\frac{1}{2}(\cancel{x'^2} + y'^2 - \cancel{2x'}y' + \cancel{x'^2} + y'^2 + 2x'\cancel{y'} - \cancel{2x'^2} + 2y'^2) - \frac{2}{\sqrt{2}}(x' - \cancel{y'} + x' + \cancel{y'}) + 1 = 0$$

$$2y'^2 = 2\sqrt{2}\,x' - 1$$

$$y'^2 = \sqrt{2}\,(x' - \frac{\sqrt{2}}{4})$$

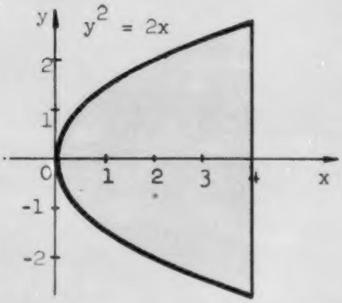
Now translating the x', y'-axes, by substituting

$$x' = x'' + \frac{\sqrt{2}}{4} \quad \text{and} \quad y' = y'',$$

we have
$$y''^2 = \sqrt{2}\, x''$$

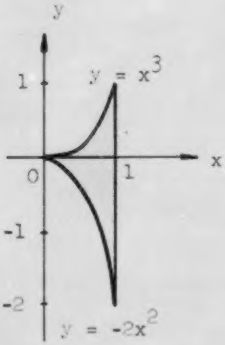
which is certainly a parabola.

5. Since $y^2 = 2x$ is symmetric with respect to the x-axis, the area of the entire region may be expressed as double that of the part above the x-axis. Therefore, we have



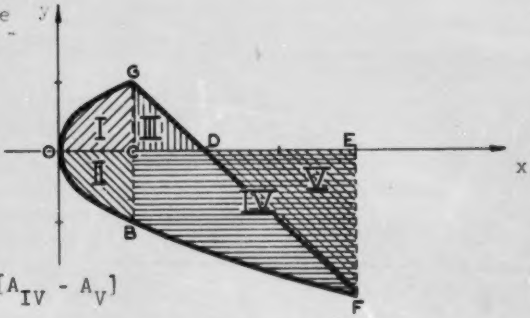
$$A = 2\int_0^4 \sqrt{2x}\, dx = 2\sqrt{2}\int_0^4 x^{1/2} dx$$
$$= 2\sqrt{2} \cdot \frac{2}{3} x^{3/2} \Big|_0^4$$
$$= \frac{4\sqrt{2}}{3}(4)^{3/2} = \frac{32\sqrt{2}}{3}$$

6. $$A = \int_0^1 x^3\, dx + \int_0^1 -(-2x^2)dx$$
$$= \int_0^1 x^3\, dx + \int_0^1 2x^2\, dx$$
$$= \int_0^1 (x^3 + 2x^2)dx = (\frac{x^4}{4} + \frac{2}{3} x^3) \Big|_0^1$$
$$= \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$$



7. (a) Subregions are defined by the following:

Region I : OCG
Region II : OCB
Region III : GCD
Region IV : BCEF
Region V : DEF



$\therefore$ Area $= A_I + A_{II} + A_{III} + [A_{IV} - A_V]$

(b) Subregions are defined by the following:

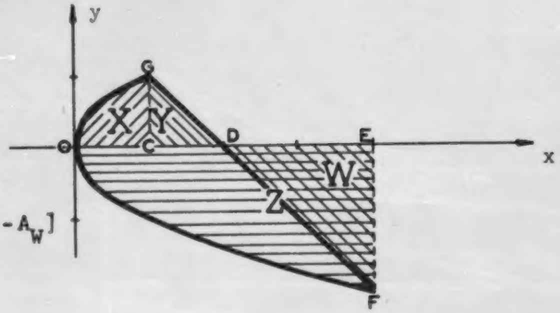
Region X: OCG

Region Y: GCD

Region Z: OEF

Region W: DEF

$\therefore$ Area $= A_X + A_Y + [A_Z - A_W]$



(c) Simplifying part (a)

$$A = \int_0^1 \sqrt{x}\,dx + \int_0^1 -(-\sqrt{x})dx + \int_1^2 (-x+2)dx + \int_1^4 -(-\sqrt{x})dx - \int_2^4 -(-x+2)dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^2 (-x+2)dx + \int_2^4 (-x+2)dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^4 (-x+2)dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x+2) + \sqrt{x}]dx$$

Simplifying part (b)

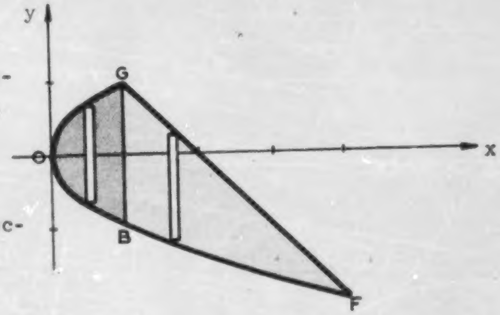
$$A = \int_0^1 \sqrt{x}\,dx + \int_1^2 (-x+2)dx + \int_0^4 -(-\sqrt{x})dx - \int_2^4 -(-x+2)dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_0^4 \sqrt{x}\,dx + \int_1^2 (-x+2)dx + \int_2^4 (-x+2)dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_0^1 \sqrt{x}\,dx + \int_1^4 \sqrt{x}\,dx + \int_1^4 (-x+2)dx$$

$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 [(-x+2) + \sqrt{x}]dx$$

In order to see the relationship between this integral expression for the area, divide the region into two parts. If we add $\underline{n}$ rectangles in region OGB, we have each rectangle with a height of $\sqrt{x} - (-\sqrt{x})$ or $(2\sqrt{x})$.



If we add $\underline{n}$ rectangles in region BEF, we have each rectangle with a height of $(-x+2) - (-\sqrt{x})$ or $(-x+2+\sqrt{x})$.

(d) $A = 2\int_0^1 \sqrt{x}\,dx + \int_1^4 (-x + 2 + \sqrt{x})dx$

$$= \frac{4}{3}x^{3/2}\Big|_0^1 + \left(-\frac{x^2}{2} + 2x + \frac{2}{3}x^{3/2}\right)\Big|_1^4$$

$$= [\frac{4}{3} - 0] + [(-8 + 8 + \frac{16}{3}) - (-\frac{1}{2} + 2 + \frac{2}{3})]$$

$$= \frac{4}{3} + \frac{16}{3} - \frac{13}{6}$$

$$= \frac{27}{6} = \frac{9}{2}$$

8. (a) (i) Area of Region I $= \int_0^2 (2x - x^2)dx$

(ii) Area of Region II $= \int_{-1}^0 -(2x - x^2)dx$

(iii) Area of Region III $= \int_2^3 -(2x - x^2)dx$

(iv) Area of Region IV $= \int_{-1}^3 -(-3)dx$

(b) Area of region bounded by $y = 2x - x^2$ and $y = -1$ can be expressed as follows:

A = Area of Region I + [Area of Region IV - Area of Region II
- Area of Region III]

$$= \int_0^2 (2x - x^2)dx + \int_{-1}^3 3\,dx + \int_{-1}^0 (2x + x^2)dx + \int_2^3 (2x - x^2)dx$$

Combining integrals 1, 3, and 4, we have

$$A = \int_{-1}^3 (2x - x^2)dx + \int_{-1}^3 3\,dx$$

$$= \int_{-1}^3 (2x - x^2 + 3)dx$$

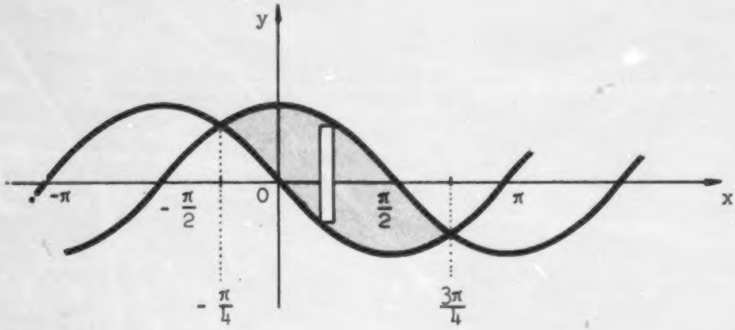
[Note that $(2x - x^2) - (-3)$ or $(2x - x^2 + 3)$ is the height of each rectangle.]

(c) $A = \int_{-1}^{3} (2x - x^2 + 3)dx$

$= (x^2 - \frac{x^3}{3} + 3x)\Big|_{-1}^{3} = (9 - 9 + 9) - (1 + \frac{1}{3} - 3) = \frac{32}{3}$

9. (a) $\begin{cases} y = \cos x \\ y = -\sin x \end{cases}$

Intersections for $|x| \leq \pi$ are $x = \frac{3\pi}{4}$ and $x = -\frac{\pi}{4}$.



$$A = \underbrace{\int_{-\pi/4}^{\pi/2} \cos x\,dx - \int_{-\pi/4}^{0} (-\sin x)dx}_{\text{area above x-axis}} + \underbrace{\int_{0}^{3\pi/4} -(-\sin x)dx - \int_{\pi/2}^{3\pi/4} (-\cos x)dx}_{\text{area below x-axis}}$$

Combining integrals, we have

$A = \int_{-\pi/4}^{3\pi/4} \cos x\,dx + \int_{-\pi/4}^{3\pi/4} \sin x\,dx = \int_{-\pi/4}^{3\pi/4} (\cos x + \sin x)dx$

[Note that height of rectangle is $\cos x - (-\sin x)$ or $(\cos x + \sin x)$.]

$\therefore A = (\sin x - \cos x)\Big|_{-\pi/4}^{3\pi/4} = [\sin \frac{3\pi}{4} - \cos \frac{3\pi}{4}] - [\sin (-\frac{\pi}{4}) - \cos(-\frac{\pi}{4}$

$= [\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}] - [-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}] = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

(b) (i)
$$\int_{-\pi/4}^{3\pi/4} \cos x \, dx = \sin x \Big|_{-\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(ii)
$$\int_{-\pi/4}^{3\pi/4} (-\sin x) dx = \cos x \Big|_{-\pi/4}^{3\pi/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

(iii)
$$\int_{-\pi/4}^{3\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_{-\pi/4}^{3\pi/4}$$
$$= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

(iv) These are signed areas. The answer to (iii) is zero which means that the area above the axis is equal to the area below the axis.

10. (a) Find $\int_{-a}^{a} f$ when f is an odd function.

Since $f(-x) = -f(x)$ the area bounded by $x = -a$, $f(x)$ and $x = 0$ has the same magnitude but opposite sign when compared to the area bounded by $x = a$, $f(x)$ and $x = 0$.

Since
$$\int_{-a}^{0} f = -\int_{0}^{a} f$$

then
$$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f$$
$$= -\int_{0}^{a} f + \int_{0}^{a} f$$
$$= 0$$

(b) When f is an even function, $f(x) = f(-x)$. The area bounded by $x = -a$, $f(x)$ and $x = 0$ is numerically equal to the area bounded by $x = a$, $f(x)$ and $f(0)$.

Since
$$\int_{-a}^{0} f = \int_{0}^{a} f$$

then
$$\int_{-a}^{a} f = \int_{-a}^{0} f + \int_{0}^{a} f = 2\int_{0}^{a} f$$

(c) $\int_{-5}^{5}(x^3 - 3x)\sin x^2\,dx$

$f : x \rightarrow (x^3 - 2x)$ is an odd function.

$g : x \rightarrow \sin x^2$ is an even function.

The product $f \cdot g$ is an odd function.

Thus $\int_{-a}^{a} f \cdot g = 0$ by part (a).

11. If $F' = f$ and $G' = g$ and $f(x) \leq g(x)$ for $a \leq x \leq b$ then $F(b) - F(a) \leq G(b) - G(a)$. By 7 - 2 - (5). If $f(x) \leq g(x)$ for $a \leq x \leq b$ then $\int_a^b f \leq \int_a^b g$. By the Fundamental Theorem of Calculus

$$\int_a^b f = F(b) - F(a)$$

and

$$\int_a^b g = G(b) - G(a).$$

Thus

$$F(b) - F(a) \leq G(b) - G(a).$$

12. $\int_a^b f = F(b) - F(a)$

Verify (5)(a): $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

Let $h(x) = f(x) + g(x)$ and $H(x) = F(x) + G(x)$.

$$\begin{aligned}\int_a^b h(x)dx &= H(b) - H(a)\\ &= (F(b) + G(b)) - (F(a) + G(a))\\ &= (F(b) - F(a)) + (G(b) - G(a))\\ &= \int_a^b f(x)dx + \int_a^b g(x)dx\end{aligned}$$

Verify (5)(b): $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

$$\int_a^b c\, f(x)dx = cF(a) - cF(b)$$

$$= c(F(a) - F(b))$$

$$= c\int_a^b f(x)dx$$

Verify (5)(c): $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a \leq c \leq b$

$$\int_a^c f(x)dx = F(c) - F(a)$$

$$\int_a^b f(x)dx = F(b) - F(c)$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = (F(c) - F(a)) + (F(b) - F(c))$$

$$= (F(b) - F(a)) + (F(c) - F(c))$$

$$= \int_a^b f(x)dx$$

13. $F(x) = \int_x^1 f$ where $f : x \to e^x$

(a) $F(1) = 0$

(b) Since $\int_a^b f = -\int_b^a f$ then $F(x) = -\int_1^x f$

$$= -(e^x - e^1)$$

$$= e - e^x$$

(c) $F'(x) = 0 - e^x = -e^x$

(d) If $G(x) = \int_x^b g$ then $G'(x) = -g(x)$

Recall that the Area Theorem stated that when

$$F(x) = \int_a^x f$$

then $$F'(x) = f(x).$$

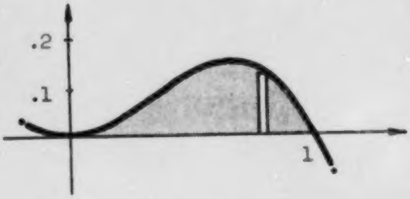
The integral $G(x) = \int_x^a g$ must be written as an integral from a to x rather than from x to a.

We have defined $\int_a^b f = -\int_b^a f$ thus $\int_x^a g = -\int_a^x g = \int_a^x -g$.

Thus $G'(x) = -g(x)$.

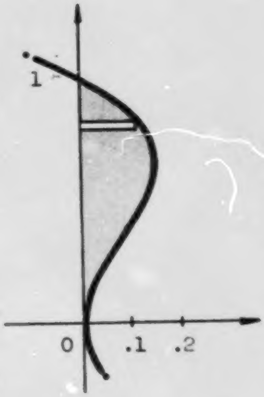
14. (a) $A = \int_0^1 (x^2 - x^3)dx = (\frac{x^3}{3} - \frac{x^4}{4}) \Big|_0^1$

Therefore $A = (\frac{1}{3} - \frac{1}{4}) - 0 = \frac{1}{12}$.



(b) It is intuitively obvious that this area is also $\frac{1}{12}$. This problem illustrates a type of problem not discussed in the text. Since it would be very difficult to set up the area integral so that the rectangles are summed along the x-axis, it is possible to sum rectangles along the y-axis. Then we would have

$$A = \int_0^1 (y^2 - y^3)dy = (\frac{y^3}{3} - \frac{y^4}{4}) \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$



Solutions Exercises 7-6

1. $\int (x^2 + 1)dx = \frac{x^3}{3} + x$

2. $\int (\frac{1}{x^2} + x + x^4)dx = -\frac{1}{x} + \frac{x^2}{2} + \frac{x^5}{5}$

3. $\int 8\sqrt{x}\, dx = 8\int x^{1/2}dx = \frac{16}{3} x^{3/2}$

4. $\int (x^2 - \sqrt{x})dx = \frac{x^3}{3} - \frac{2x^{3/2}}{3}$

5. $\int (\frac{1 - x}{x})dx = \int (\frac{1}{x} - 1)dx = \log_e x - x$

6. $\int \sin 3x\, dx = -\frac{1}{3} \cos 3x$

7. $\int \cos(2x - 5)dx = \frac{1}{2} \sin(2x - 5)$

8. $\int (-\sin 2x)dx = -\int \sin 2x\, dx = -(-\frac{1}{2} \cos 2x) = \frac{1}{2} \cos 2x$

9. $\int [-\cos(3x - 1)]dx = -\int \cos(3x - 1)dx = -\frac{1}{3} \sin(3x - 1)$

10. $\int \frac{4}{3} \cos 3x\, dx = \frac{4}{3}\int \cos 3x\, dx = \frac{4}{9} \sin 3x$

11. $\int 2 \sin x \cos x\, dx = \int \sin 2x = -\frac{1}{2} \cos 2x$

12. $\int (3 \sin 2x - 6 \cos 3x)dx = -\frac{3}{2} \cos 2x - 2 \sin 3x$

13. $\int e^{2x}\, dx = \frac{1}{2} e^{2x}$

14. $\int e^{x/3}\, dx = 3e^{x/3}$

15. $\int (e^x + e^{-x})^2\, dx = \int (e^{2x} + 2 + e^{-2x})dx = \frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x}$

16. $$\int x^2e^x dx = x^2e^x - 2\int xe^x dx$$
$$= x^2e^x - 2(xe^x - e^x)$$
$$= x^2e^x - 2xe^x + 2e^x$$
$$= e^x(x^2 - 2x + 2)$$

17. $$\int x^3e^x dx = x^3e^x - 3\int x^2e^x dx$$
$$= x^3e^x - 3[e^x(x^2 - 2x + 2)]$$
$$= e^x(x^3 - 3x^2 + 6x - 6)$$

18. $$\int x^4e^x dx = x^4e^x - 4\int x^3e^x dx$$
$$= x^4e^x - 4[e^x(x^3 - 3x^2 + 6x - 6)]$$
$$= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$$

19. $$\int x^2 \log_e x \, dx = \frac{x^3}{3}(\log_e x - \frac{1}{3})$$

20. $$\int x^3 \log_e x \, dx = \frac{x^4}{4}(\log_e x - \frac{1}{4})$$

21. $$\int x^4 \log_e x \, dx = \frac{x^5}{5}(\log_e x - \frac{1}{5})$$

22. $$\int x^2 \sin x \, dx = -x^2 \cos x + 2\int x \cos x \, dx$$
$$= -x^2 \cos x + 2[x \sin x + \cos x]$$
$$= 2x \sin x + (2 - x^2)\cos x$$

23. $$\int x^3 \sin x \, dx = -x^3 \cos x + 3\int x^2 \cos x \, dx$$
$$= -x^3 \cos x + 3[x^2 \sin x - 2\int x \sin x \, dx]$$
$$= -x^3 \cos x + 3[x^2 \sin x - 2(-x \cos x + \sin x)]$$
$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$$
$$= (6x - x^3)\cos x + (3x^2 - 6)\sin x$$

24. $$\int e^{3x} \sin 4x \, dx = \frac{e^{3x}}{25}(3 \sin 4x - 4 \cos 4x)$$

25. $\int e^{x/2} \cos \frac{3x}{2}\, dx = \frac{e^{x/2}}{\frac{10}{4}}(\frac{3}{2} \sin \frac{3x}{2} + \frac{1}{2} \cos \frac{3x}{2})$

$$= \frac{2}{5} e^{x/2} \cdot \frac{1}{2} (3 \sin \frac{3x}{2} + \cos \frac{3x}{2})$$

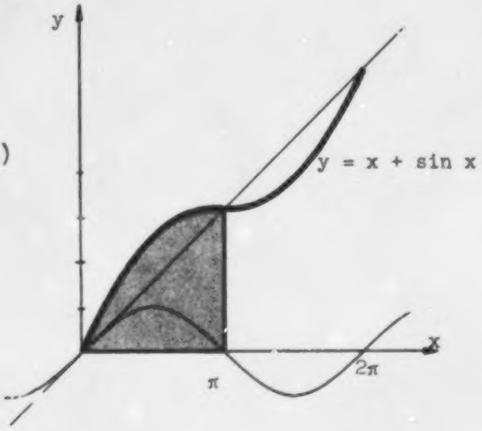
$$= \frac{1}{5} e^{x/2}(3 \sin \frac{3x}{2} + \cos \frac{3x}{2})$$

26. $\int_0^{\pi} (x + \sin x)dx$

$$= (\frac{x^2}{2} - \cos x) \Big|_0^{\pi}$$

$$= (\frac{\pi^2}{2} + 1) - (0 - 1)$$

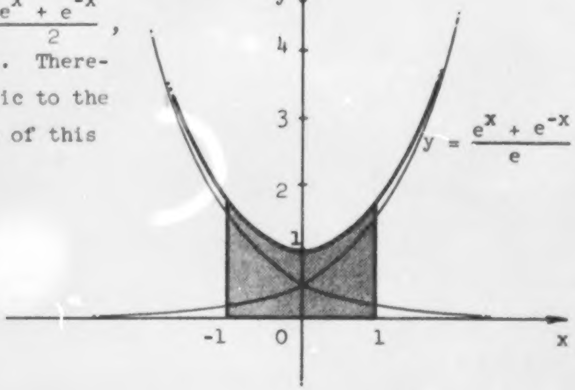
$$= \frac{\pi^2}{2} + 2 \approx 6.9$$



27. $\int_0^{2\pi} (x + \sin x)dx = (\frac{x^2}{2} - \cos x) \Big|_0^{2\pi} = (2\pi^2 - 1) - (0 - 1) = 2\pi^2 \approx 2.35$

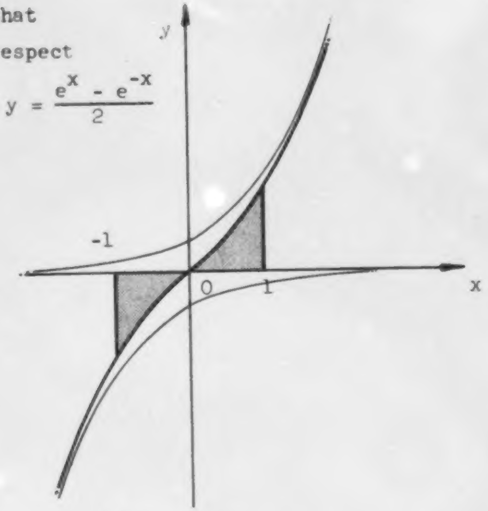
28. $\int_{-1}^{1} \frac{e^x + e^{-x}}{2}\, dx$

This function $x \rightarrow \frac{e^x + e^{-x}}{2}$, is an even function. Therefore, it is symmetric to the y-axis. Making use of this symmetry, we have

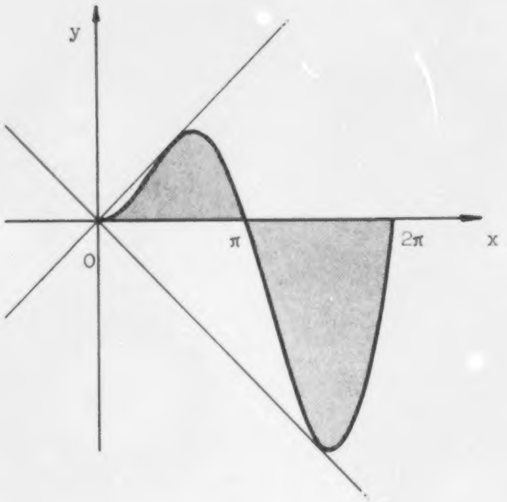


$$\int_{-1}^{1} \frac{e^x + e^{-x}}{2} = 2[\frac{1}{2}]\int_0^1 (e^x + e^{-x})dx = (e^x - e^{-x}) \Big|_0^1 = e - \frac{1}{e} \approx 2.35$$

29. (a) $\int_{-1}^{1} \frac{e^{x} - e^{-x}}{2} = 0$. Here we have an odd function which means that the curve is symmetric with respect to the origin.


y
$y = \frac{e^{x} - e^{-x}}{2}$
-1
0
1
x

30. (a) $$\int_{0}^{2\pi} x \sin x \, dx = (-x \cos x + \sin x) \Big|_{0}^{2\pi}$$
$$= (-2\pi + 0) - (0 + 0)$$
$$= -2\pi$$



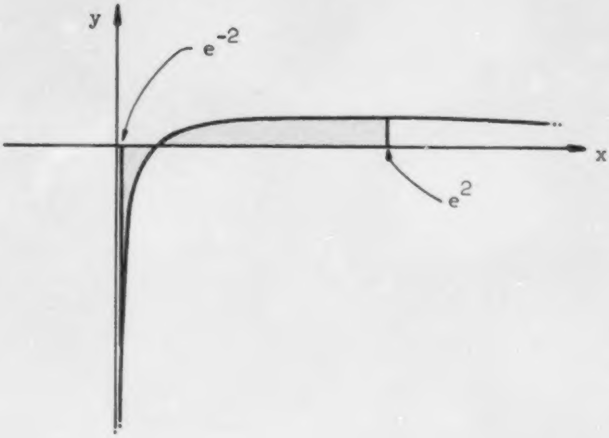
31. (a) $$\int_{1/e^2}^{e^2} \frac{\log_e x}{\sqrt{x}}\,dx = \int_{1/e^2}^{e^2} x^{(-1/2)}\log_e x\,dx$$

$$= \frac{x^{1/2}}{\frac{1}{2}}\left(\log_e x - \frac{1}{\frac{1}{2}}\right)\Bigg|_{1/e^2}^{e^2}$$

$$= 2\sqrt{x}(\log_e x - 2)\Bigg|_{1/e^2}^{e^2}$$

$$= [2e(2 - 2)] - [\tfrac{2}{e}(-2 - 2)]$$

$$= 0 + \frac{8}{e} = \frac{8}{e} \approx 2.96$$

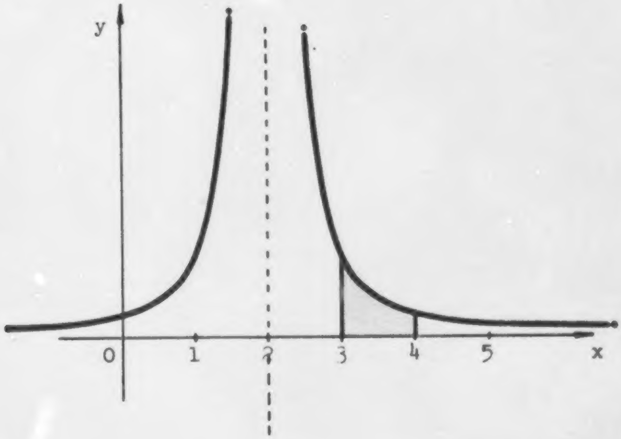


32. $$A = \int_3^4 \frac{1}{(x - 2)^2}\,dx$$

$$= -(x - 2)^{-1}\Bigg|_3^4$$

$$= -\frac{1}{2} + 1$$

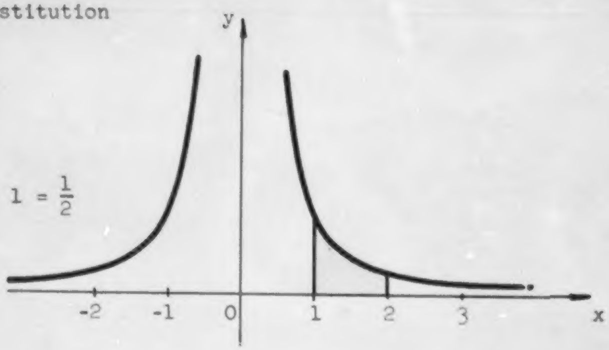
$$= \frac{1}{2}$$



Replace x - 2 by x (i.e., x by x + 2). This linear substitution leads to

$$A_{L.S.} = \int_1^2 \frac{1}{x^2}\,dx$$

$$= -x^{-1}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$



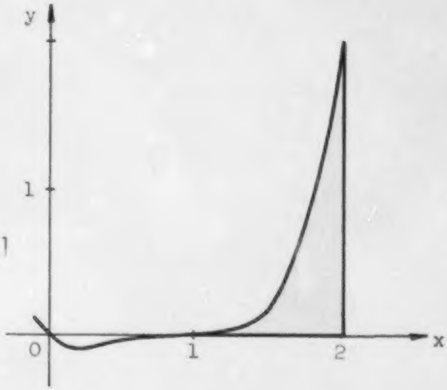
33. $$A = \int_1^2 x(x-1)^3$$

$$= \int_1^2 (x^4 - 3x^3 + 3x^2 - x)dx$$

$$= \left(\frac{x^5}{5} - \frac{3}{4}x^4 + x^3 - \frac{x^2}{2}\right)\Big|_1^2$$

$$= [\frac{32}{5} - 12 + 8 - 2] - [\frac{1}{5} - \frac{3}{4} + 1 - \frac{1}{2}]$$

$$= [\frac{2}{5}] + [\frac{1}{20}] = \frac{9}{20}$$

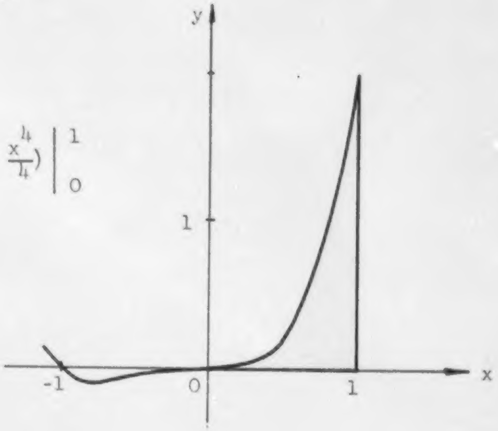


Replace x - 1 by x (i.e., x by x + 1). This linear substitution leads to

$$A_{L.S.} = \int_0^1 (x+1)x^3\,dx$$

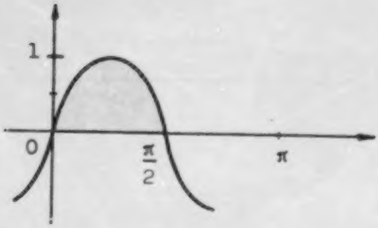
$$= \int_0^1 (x^4 + x^3)dx = \left(\frac{x^5}{5} + \frac{x^4}{4}\right)\Big|_0^1$$

$$= \frac{1}{5} + \frac{1}{4} = \frac{9}{20}$$



34. $A = \int_0^{\pi/2} \sin 2x \, dx$

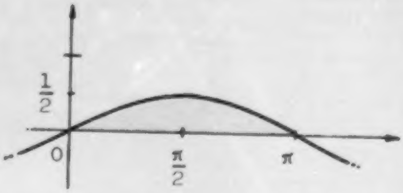
$= -\frac{1}{2} \cos 2x \Big|_0^{\pi/2} = 1$



Substitute x for 2x (i.e., let $x = \frac{x}{2}$)

$\therefore A_{L.S.} = \frac{1}{2}\int_0^{\pi} \sin x \, dx$

$= -\frac{1}{2} \cos x \Big|_0^{\pi} = 1$

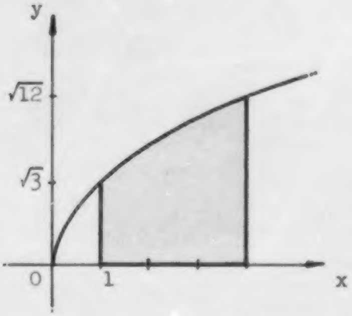


35. $A = \int_1^4 \sqrt{3x} \, dx$

$= \sqrt{3}\int_1^4 x^{1/2} \, dx$

$= \sqrt{3} \cdot \frac{2}{3} x^{3/2} \Big|_1^4$

$= \frac{2\sqrt{3}}{3}(8 - 1) = \frac{14}{3} \sqrt{3}$

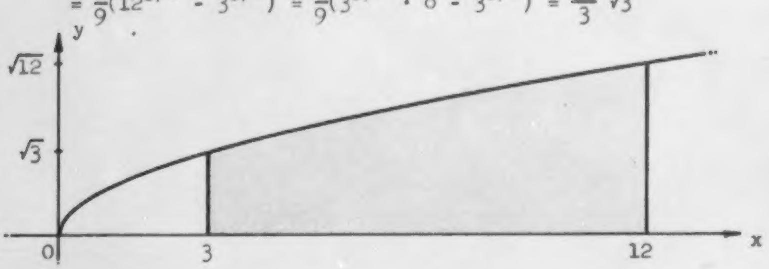


Substitute x for 3x (i.e., let $x = \frac{x}{3}$).

$\therefore A_{L.S.} = \frac{1}{3}\int_3^{12} \sqrt{x} \, dx$

$= \frac{1}{3} \cdot \frac{2}{3} x^{3/2} \Big|_3^{12}$

$= \frac{2}{9}(12^{3/2} - 3^{3/2}) = \frac{2}{9}(3^{3/2} \cdot 8 - 3^{3/2}) = \frac{14}{3} \sqrt{3}$



36. (a) An intuitive treatment of this problem leads one to compare the graphs of

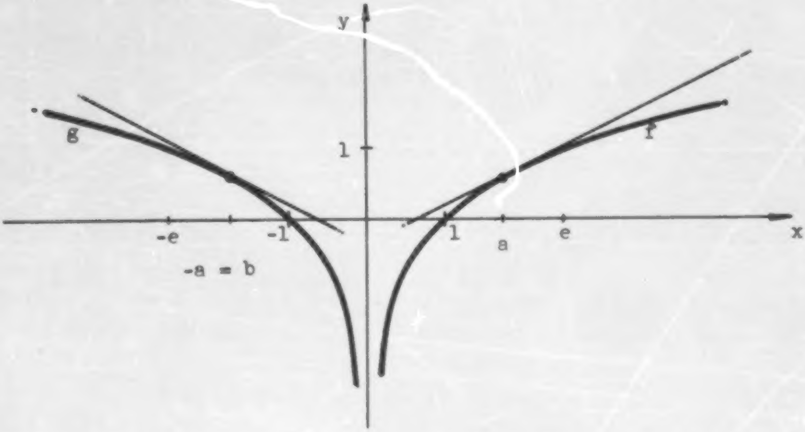
$$f : x \to \log_e(x)\ ,\ x > 0$$

and

$$g : x \to \log_e(-x)\ ,\ x < 0.$$

The tangent line to f at $a > 0$ has the slope $D \log_e a = \frac{1}{a}$.

Because of symmetry the tangent line to g at $b = -a$ is the opposite of the slope of f at $a > 0$. Thus the slope of g at b is $\frac{1}{-a} = \frac{1}{b}$. It follows then that $D \log_e(-x) = \frac{1}{x}$.



If we look back at 6 - 6 - (13) we can obtain the same results directly.

$$D \log_a(cx + d) = \frac{c}{(cx + d)\log_e a}$$

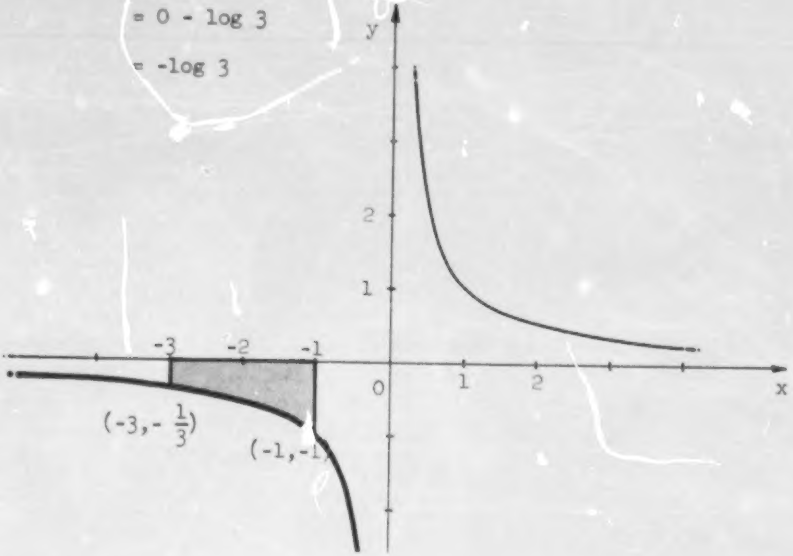
Let $c = -1$, $d = 0$ and $a = e$

$$D \log_e(-x) = \frac{-1}{-x} = \frac{1}{x}\ .$$

(b) $\int_{-3}^{-1} \frac{1}{x}\,dx = \log_e(-(-1)) - \log(-(-3))$

$= 0 - \log 3$

$= -\log 3$



37. (a) By the Fundamental Theorem of Calculus f must have no gaps on the interval [a,b]. In the case of $f : x \rightarrow \frac{1}{x}$, f on the interval [-1,1] has a gap at x = 0. Thus we cannot apply the Fundamental Theorem of Calculus.

(b) $\lim_{n \to \infty} \int_{1/n}^{1} \frac{1}{x}\,dx = \lim_{n \to \infty} (\log_e 1 - \log_e \frac{1}{n})$

$= \lim_{n \to \infty} (\log_e 1 + \log_e n)$

$= 0 + \infty$

$= \infty$

(c) The area assigned to the region bounded by $y = \frac{1}{x}$, $x \neq 0$, the y-axis, the x-axis and x = 1 is well defined so long as $x \neq 0$. This area has a finite value so long as x has a finite value. Should we take an integral with x = 0 as an endpoint then the integral is undefined.

(d) Since f has a gap on the interval $-1 \leq x \leq 1$ we cannot apply the Fundamental Theorem of Calculus. We can, however, try the mechanics of integration. We find that our answer is zero.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x}\,dx &= \log(1) - \log(-(-1)) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

This might be consistent to the extent that we are integrating over two areas with the "same" magnitude but opposite in sign. Thus they cancel each other out.

On the other hand the inconsistency comes in our effort to write the statement needed to justify cancelling out areas, namely:

$$\infty + (-\infty) = 0.$$

Chapter 8

DIFFERENTIATION THEORY AND TECHNIQUE

Solutions Exercises 8-1

1. (a) (i)

$$y = x^3 - 3x + 3$$

$$y + \Delta y = (x + \Delta x)^3 - 3(x + \Delta x) + 3$$

$$= x^3 + 3x^2\,\Delta x + 3x\,\Delta x^2 + \Delta x^3 - 3x - 3\Delta x + 3$$

$$\Delta y = (3x^2 - 3)\Delta x + 3x\,\Delta x^2 + \Delta x^3$$

$$\frac{\Delta y}{\Delta x} = 3x^2 - 3 + 3x\,\Delta x + \Delta x^2$$

(ii) $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 3x^2 - 3$ or $y' = 3x^2 - 3$

(b) (i)

$$y = \sqrt{x}$$

$$y + \Delta y = \sqrt{x + \Delta x}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

(ii) $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{1}{2\sqrt{x}}$ or $y' = \dfrac{1}{2\sqrt{x}}$

(c) (i)

$$y = 3x^2 - \frac{1}{x}$$

$$y + \Delta y = 3(x + \Delta x)^2 - \frac{1}{x + \Delta x}$$

$$\Delta y = (3x^2 + 6x\,\Delta x + 3\Delta x^2 - 3x^2) + (\frac{1}{x} - \frac{1}{x + \Delta x})$$

$$= 6x\,\Delta x + 3\Delta x^2 + \frac{x + \Delta x - x}{x(x + \Delta x)}$$

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x + \frac{1}{x(x + \Delta x)}$$

(ii) $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 6x + \dfrac{1}{x^2}$ or $y' = 6x + \dfrac{1}{x^2}$

2. (a) $y = e^{cx}$

$$\frac{dy}{dx} = ce^{cx}$$

$$\frac{d^2y}{dx^2} = c^2\,e^{cx}$$

(b) $y = -\log_e x$ -- Show that $\frac{dy^2}{dx^2} = (\frac{dy}{dx})^2$.

$$\frac{dy}{dx} = -\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{1}{-x^2}\right)$$

$$= \frac{1}{x^2} = (\frac{dy}{dx})^2$$

(c) The result $\frac{d^2y}{dx^2} = (\frac{dy}{dx})^2$ certainly holds for $y = -\log_e x + c$.

Otherwise it is not a general result.

3. $f(x) \approx g(x) = 1 + 3(x - 2)$ in the sense that $\lim\limits_{x \to 2} \frac{f(x) - g(x)}{x - 2} = 0$.

Near $x = 2$, $f(x) \approx g(x)$. Thus $f(2) \approx g(2) = 1$.
On the other hand $f'(2)$ need not equal $g'(2)$. In fact $f'(2)$ might not even be defined. Take, for example, $f(x) = 1 + |3(x - 2)|$. In this case there is no $f'(2)$, even though $f(2) = g(2)$.

4. (a) $$\lim_{\Delta x \to 0} \frac{\left[\frac{1}{(x + \Delta x)^2} + 3\right] - \left[\frac{1}{x^2} + 3\right]}{\Delta x} = D\left(\frac{1}{x^2} + 3\right) = \frac{-2}{x^3}$$

(b) $$\lim_{\Delta x \to 0} \frac{\left((x + \Delta x) + \frac{1}{(x + \Delta x)}\right)^2 - (x + \frac{1}{x})^2}{\Delta x} = D(x + \frac{1}{x})^2$$

$$= \left(D\, x^2 + 2 + \frac{1}{x^2}\right)$$

$$= 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^4 - 1)$$

(c) $$\lim_{\Delta x \to 0} \frac{(x + \Delta x)\sqrt{x + \Delta x} - x\sqrt{x}}{\Delta x} = D(x^{3/2}) = \frac{3}{2}\sqrt{x}$$

5. (a) $\lim\limits_{h \to 0} \frac{(3 + h)^2 - (3)^2}{h} = f'(3)$, where $f(x) = x^2$

$\therefore$ $f'(x) = 2x$ and $f'(3) = 6$

(b) $\lim\limits_{\Delta x \to 0} \frac{(4 + \Delta x)^{3/2} - 4^{3/2}}{\Delta x} = f'(4)$, where $f(x) = x^{3/2}$

$\therefore$ $f'(x) = \frac{3}{2}\sqrt{x}$ and $f'(4) = 3$

(c) $\lim\limits_{h \to 0} \frac{1 - (1 + h)^6}{12h} = \lim\limits_{h \to 0} - \frac{\frac{1}{12}(1 + h)^6 - \frac{1}{12}(1)^6}{h} = -f'(1)$ where $f(x) = \frac{x^6}{12}$

$\therefore\ f'(x) = \frac{x^5}{2}$ and $f'(1) = \frac{1}{2}$

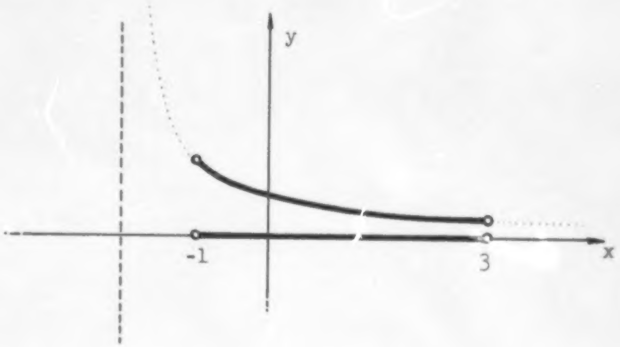
(d) $\lim\limits_{\Delta x \to 0} \frac{\pi \sin (\pi + h) - \pi \sin \pi}{h} = f'(\pi)$ where $f(x) = \pi \sin x$

$\therefore\ f'(x) = \pi \cos x$ and $f'(\pi) = -\pi$

(e) $\lim\limits_{\Delta x \to 0} \frac{3 \cos [\pi - 2(\frac{\pi}{8}) + 2\Delta x)] - 3 \cos (\pi - 2(\frac{\pi}{8}))}{8\Delta x} = f'(\frac{\pi}{8})$ where $f(x) = \frac{3}{8} \cos (\pi - 2x)$

$\therefore\ f'(x) = \frac{3}{4} \sin (\pi - 2x)$ and $f'(\frac{\pi}{8}) = \frac{3}{4} \sin (\frac{3\pi}{4}) = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

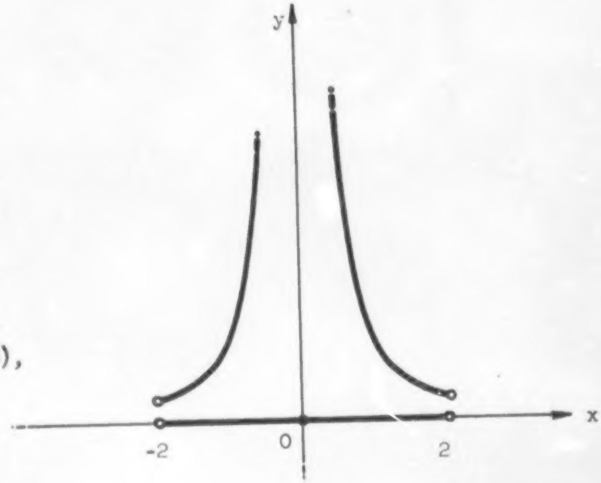
6. (a) f is continuous for all x on the interval $-1 < x < 3$.



(b) f is discontinuous at $x = 0$.

(1) First criteria is satisfied; i.e., $f(0)$ is defined.

(2) But the second criteria is not satisfied; i.e., $\lim\limits_{x \to 0} f(x) \neq f(0)$, or $\lim\limits_{x \to 0} \frac{1}{x^2} \neq 0$.



(c) f is continuous at $x = 1$, because

(1) $f(1)$ is defined to be $\frac{1}{2}$ and

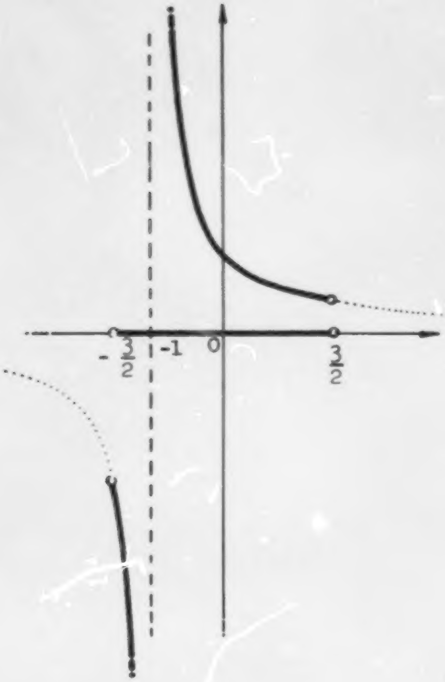
(2) $\lim\limits_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim\limits_{x \to 1} \frac{1}{x + 1} = \frac{1}{2} = f(1)$.

f is discontinuous at $x = -1$, because

(1) $f(-1)$ is not defined, and

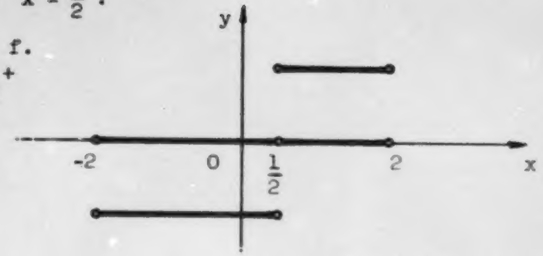
(2) $\lim\limits_{x \to -1} f(x) \neq$ a number.

∴ The function f is said to be discontinuous on the interval.

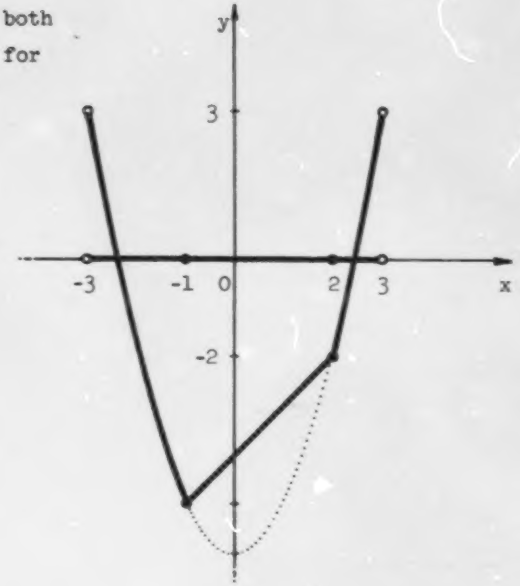


(d) f is discontinuous at $x = \frac{1}{2}$.

Since $\lim_{x \to \frac{1}{2}^-} f \neq \lim_{x \to \frac{1}{2}^+} f$.



(e) f is continuous since both criteria are satisfied for $x = -1$, and $x = 2$.

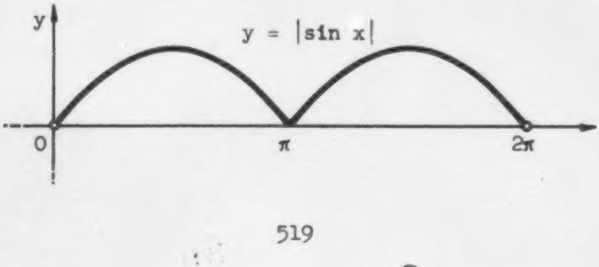


7. (a) $f : x \to |\sin x|$, i.e., $f : x \to \begin{cases} \sin x & \text{for } \sin x \geq 0 \\ -\sin x & \text{for } \sin x < 0 \end{cases}$

$\therefore \; f': x \to \begin{cases} \cos x & \text{for } 0 \leq x \leq \pi \\ -\cos x & \text{for } \pi < x \leq 2\pi \end{cases}$

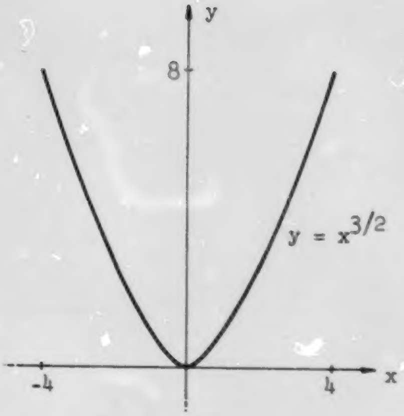
f is continuous for all x.

f is not differentiable at $x = \pi$, since $\lim_{x \to \pi^-} f' \neq \lim_{x \to \pi^+} f'$.



(b) $f : x \to x^{3/2}$, f continuous for all x.

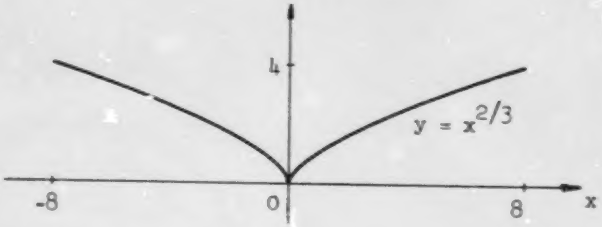
$f' : x = \frac{3}{2}\sqrt{x}$, f differentiable for all x.



(c) $f : x \to x^{2/3}$, f continuous for all x.

$f' : x \to \frac{2}{3\sqrt[3]{x}}$, f not differentiable at $x = 0$.

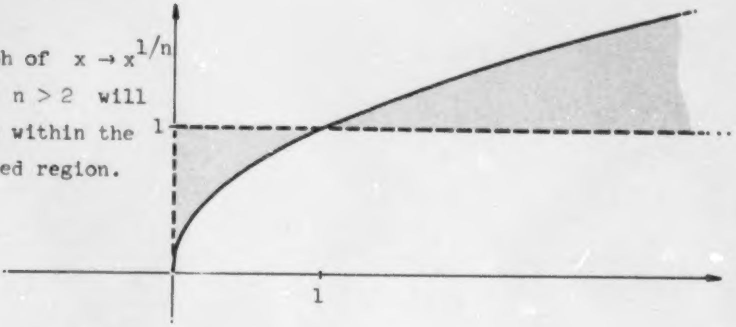
At $x = 0$, a vertical tangent line exists.



(d) $f : x \to x^{1/n}$, n even, $n \geq 2$, f continuous for all x in interval.

$f' : x \to \frac{1}{nx^{1-1/n}}$, f not differentiable at $x = 0$.

At $x = 0$, a vertical tangent line exists.

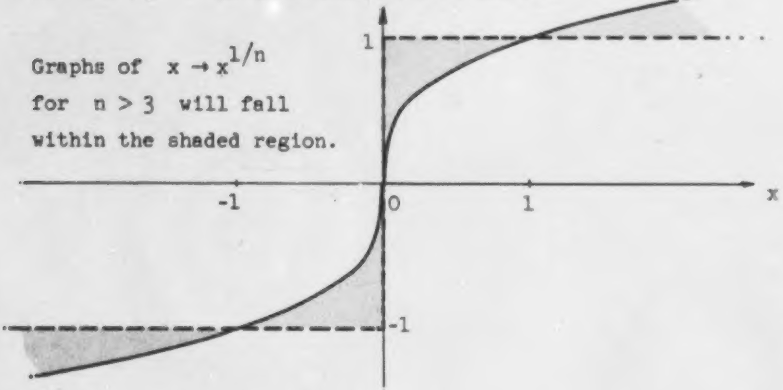


Graph of $x \to x^{1/n}$ for $n > 2$ will fall within the shaded region.

(e) $f : x \to x^{1/n}$, n odd, $n \geq 3$, f continuous for all x.

$f' : x \to \dfrac{1}{nx^{1-1/n}}$, f not differentiable at $x = 0$.

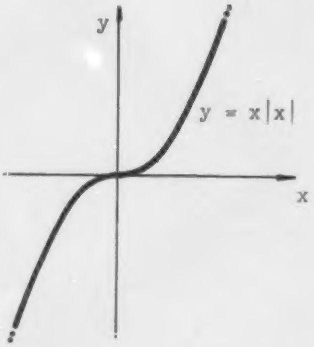
At $x = 0$, a vertical tangent line exists



Graphs of $x \to x^{1/n}$ for $n > 3$ will fall within the shaded region.

(f) $f : x \to x|x|$, f continuous for x.

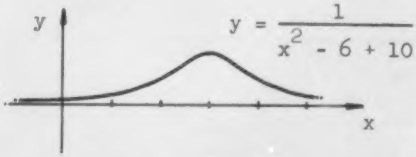
f differentiable for all x.



$$\text{i.e.,}\quad f : x \to \begin{cases} x^2 & , x \geq 0 \\ -x^2 & , x < 0 \end{cases}$$

$$f' : x \to \begin{cases} 2x & , x \geq 0 \\ -2x & , x < 0 \end{cases}$$

(g) f continuous for all x.
f differentiable for all x.



8. (a) $|1 - \cos x| \leq \frac{x^2}{2}$

$$-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$$

$$-1 - \frac{x^2}{2} \leq -\cos x \leq -1 + \frac{x^2}{2}$$

$$1 + \frac{x^2}{2} \geq \cos x \geq 1 - \frac{x^2}{2}$$

$$\lim_{x \to 0} 1 + \frac{x^2}{2} = 1$$

$$\lim_{x \to 0} 1 - \frac{x^2}{2} = 1$$

Thus $1 \leq \lim_{x \to 0} \cos x \leq 1$ and $\lim_{x \to 0} \cos x = 1$.

(b) $|\sin x| \leq |x|$

$$-|x| \leq \sin x \leq |x|$$

$$\lim_{x \to 0} -|x| = 0$$

$$\lim_{x \to 0} |x| = 0$$

Thus, $0 \leq \lim_{x \to 0} \sin x \leq 0$ and $\lim_{x \to 0} \sin x = 0$.

(c) For the sine function to be continuous everywhere, that is at every point a, it must satisfy two conditions,

(1) $\sin(a)$ must exist, and

(2) $\lim_{x \to a} \sin(x) = \sin(a)$.

As for (1), $\sin(a)$ exists for all real values of a.

When $a = 0$ we found in part (b) $\lim_{x \to 0} \sin(x) = 0 = \sin 0$. For some $a \neq 0$ we must find

$$\lim_{x \to a} \sin(a) = \lim_{h \to 0} \sin(a + h), \quad \text{where } |x - a| = h.$$

$$= \lim_{h \to 0} \sin a(\cos h) + \lim_{h \to 0} \cos a(\sin h)$$

$$= \sin a(1) + \cos a(0)$$

$$= \sin a$$

This satisfies condition (2).

9. $f : x \to \sin \frac{1}{x}, \quad x > 0$

(a) n is a positive integer.

(i) $f(\frac{1}{n\pi}) = \sin n\pi = 0$

(ii) $$f(\frac{2}{(4n + 1)\pi}) = \sin (\frac{4n + 1}{2} \pi)$$

$$= \sin(2n\pi + \frac{\pi}{2})$$

$$= 1$$

(iii) $$f(\frac{2}{4n + 3} \pi) = \sin(\frac{4n + 3}{2} \pi)$$

$$= \sin(2n\pi + \frac{3}{2} \pi)$$

$$= -1$$

(b) As $n \to \infty$ the limits are respectively

(i) 0
(ii) 1
(iii) -1

(c) The most specific statement that we can make concerning $\lim_{x \to 0} f(x)$ is $-1 \leq \lim_{x \to 0} f(x) \leq 1$. But $f(0)$ cannot be defined as an interval. Thus, there is no way to define $f(0)$ so that

$$\lim_{x \to 0} f(x) = f(0).$$

Solutions Exercises 8-2a

1. This is an intuitive application of the Intermediate Value Theorem. There were times when the motorist was going zero and at other times 70. Each speed on this interval from zero to 70 corresponds to at least one instant of time. For this example speed was a function of time.

2. (a) Consider the distance traveled to be a function of time, say $f(t) = s$. The change of distance with respect to time is speed. That is $f'(t) = v$.

   Since $f(0) = 0$ and $f(4) = 200$ by the Mean Value Theorem there exists some t_0, in the interval $0 \leq t_0 \leq 4$, such that

$$\frac{f(4) - f(0)}{4 - 0} = f'(t_0)$$

$$\frac{200 - 0}{4} = 50 = f'(t_0).$$

   Thus the speed was 50 at least once.

   (b) The acceleration was zero during time intervals when constant speeds were maintained and at the time each local maximum or minimum was reached.

3. Consider a function which associates horizontal distance to elevation; say $h(x)$. We see that $h(0) = 200'$ and $h(100) = 5480'$. By the Mean Value Theorem there is an x_0 in the interval $0 \leq x_0 \leq 200$ such that

$$\frac{h(100) - h(0)}{100 - 0} = h'(x_0)$$

or

$$\frac{5480 - 200}{100} = \frac{1 \text{ mile}}{100 \text{ miles}} = 1\%$$

   Since $h'(x_0)$ is slope, at least once on the trip the slope was precisely 1%.

4. Since acceleration is the first derivative of velocity, if acceleration is negative then by Theorem 8-2e the velocity is strictly decreasing.

5. If f'' is nonnegative then c is a minimum.
   If f'' is nonpositive then c is a maximum.

6. (a) If f is continuous at c then c is a minimum. But $f(c)$ might be undefined or otherwise discontinuous so that $f(c)$ might not be the minimum value of the function.

   (b) If f is continuous at c then c is a maximum. But $f(c)$ might be undefined or discontinuous.

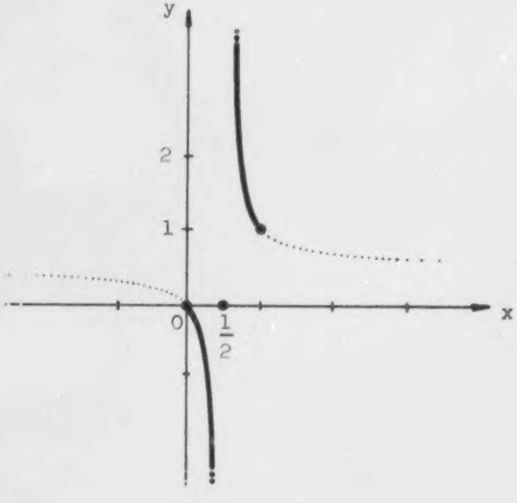
   (c) Neither a maximum nor a minimum occures at c if f is continuous at c. Otherwise it is unknown.

   (d) Neither a maximum nor a minimum occur on the interval.

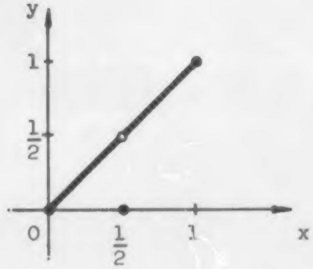
7. If f' is positive in the interval $a \leq x \leq b$ then f is a strictly increasing function by Theorem 8-2e. If f has a zero in the interval $a \leq x \leq b$, then it will be unique. The maximum occurs when $x = b$ and the minimum occurs when $x = a$.

8. (a) A few possible functions are suggested

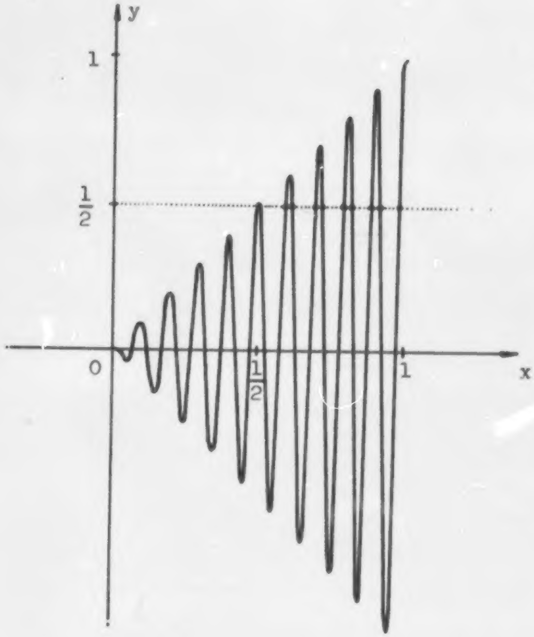
   (i) $$f : x \to \begin{cases} \dfrac{x}{2x - 1}, & x \neq \dfrac{1}{2} \\ 0, & x = \dfrac{1}{2} \end{cases}$$



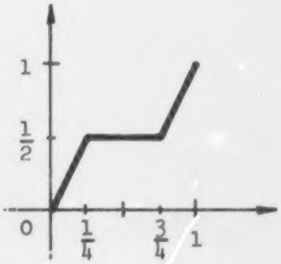
(ii) $h : x \to \begin{cases} y = x, & x \neq \frac{1}{2} \\ y = 0, & x = \frac{1}{2} \end{cases}$



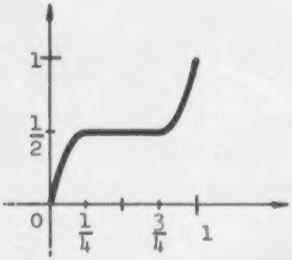
(b) (i) $h : x \to x \cos(n \pi x)$, n is an even integer, $n \geq 20$.



(ii) $f : x \to \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{4} \\ \frac{1}{2}, & \text{if } \frac{1}{4} \leq x < \frac{3}{x} \\ 2x - 1, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases}$



(iii) $f : x \to \begin{cases} -8(x - \frac{1}{4})^2 + \frac{1}{2}, & \text{if } 0 \le x \le \frac{1}{4} \\ \frac{1}{2}, & \text{if } \frac{1}{4} \le x < \frac{3}{4} \\ 8(x - \frac{3}{4})^2 + \frac{1}{2}, & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$



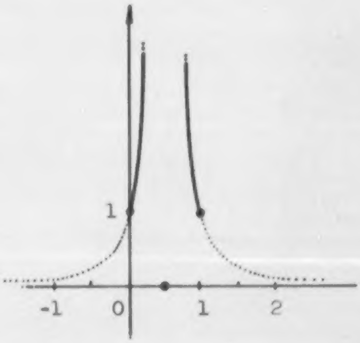
9. f is continuous, $a \le x \le b$, $f(a) \ne f(b)$ and $f(a) < d < f(b)$.

There can exist many values c such that $f(c) = d$. As one example, take the oscillating function $f : x \to \sin \frac{1}{x}$ for $x \ne 0$. Let $a = \frac{1}{200\pi}$ and $b = 1$. There could be as many as 199 values of c. In this case $f(a) = 0$ and $f(b) = \sin 1 \approx .8415$. Suppose $d = .5$, then 199 values of c yield $f(c) = .5$.

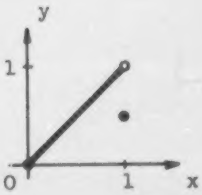
10. f is defined on $0 \le x \le 1$.

(a) f has a minimum but no maximum.

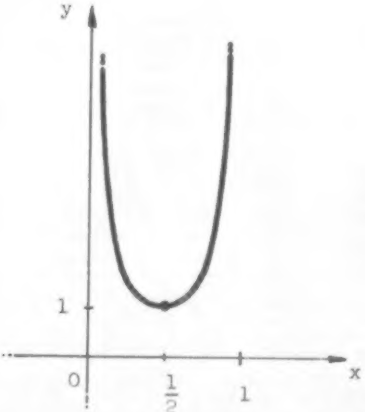
(i) $f : x \to \begin{cases} (\frac{1}{2x - 1})^2, & \text{if } x \ne \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \end{cases}$



(ii) $f : x \to \begin{cases} x & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$

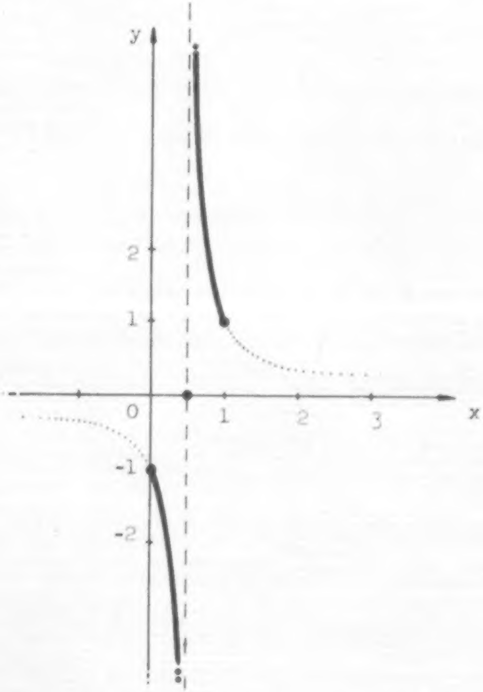


(iii) $f : x \to \csc \pi x$, if $x \neq 1$

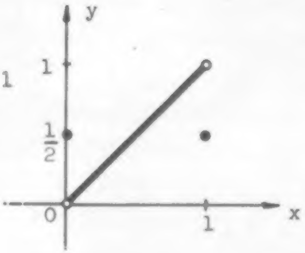


(b) f has neither a maximum or minimum.

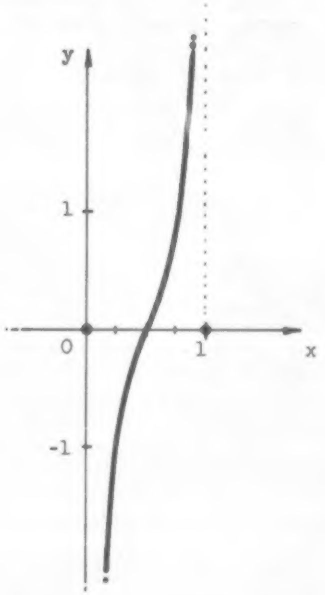
(i) $f : x \to \begin{cases} \frac{1}{2x - 1}, & \text{if } x \neq \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \end{cases}$



(ii) $f : x \to \begin{cases} x\ , & \text{if } \ 0 < x < 1 \\ \frac{1}{2}\ , & \text{if } \ x = 0 \ \text{ or } \ x = 1 \end{cases}$



(iii) $f : x \to \begin{cases} \tan \pi(x - \frac{1}{2}), & \text{if } \ x \neq 0 \ \text{ and } \ x \neq 1 \\ 0 \qquad\quad , & \text{if } \ x = 0 \ \text{ or } \ x = 1. \end{cases}$



(c) No. If f is continuous on a closed interval then there must be at least one maximum and one minimum by Theorem 8-2b.

11. $f : x \to x^2 + 1$ and $F : x \to \int_0^x f'$

(a) $F(x) = x^2$

(b) $F(x) = f(x) - f(0)$

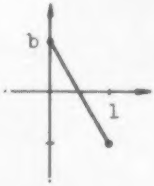
(c) If $f'(x) = 2x > 0$, then $\int_a^b f' \geq 0$ on the interval $0 \leq a \leq x \leq b$.

By Theorem 12-2d $\int_a^b f' = f(b) - f(a) \geq 0$ or $f(a) \leq f(b)$ when $a \leq b$ and f is increasing.

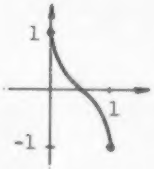
12. f is continuous on $0 \leq x \leq 1$ and maximum at $f(c)$.
A number of possible examples are presented.

(a) $c = 0$, $f'(0) \neq 0$

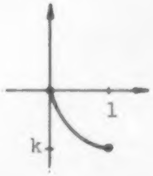
(i) $f : x \rightarrow mx + b$, $m < 0$



(ii) $f : x \rightarrow \cot(\frac{\pi}{2} x + \frac{\pi}{4})$

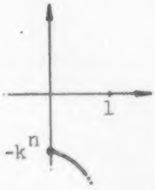


(iii) $f : x \rightarrow kx^n$, $k < 0$, $n > 0$

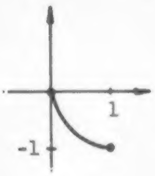


(iv) $f : x \rightarrow \dfrac{1}{(x - k)^n}$, $k > 1$

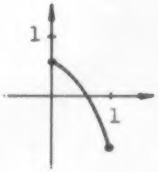
n is an odd positive integer.



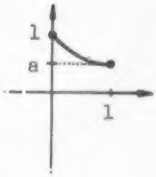
(v) $f : x \to -\sin \frac{\pi}{2} x$



(vi) $f : x \to a(x - h)^2 + k, \; a < 0$
$h < 0$

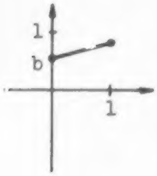


(vii) $f : x \to a^x, \; 0 < a < 1$

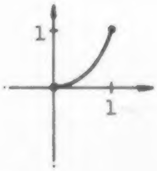


(b) $c = 1, \; f'(1) \neq 0$

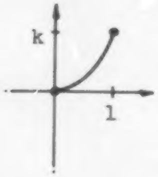
(i) $f : x \to mx + b, \; m > 0$



(ii) $f : x \to \tan \frac{\pi}{4} x$

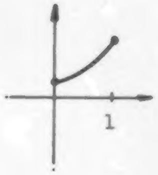


(iii) $f : x \to k x^n$, $k > 0, n > 0$

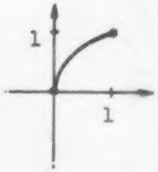


(iv) $f : x \to \frac{1}{(x - k)^n}$, $k > 1$

$k > 1$, n is an even positive integer.

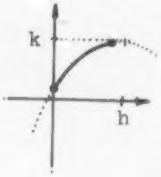


(v) $f : x \to \sin \frac{\pi}{2} x$



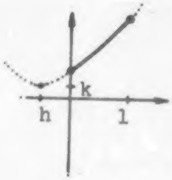
(vi) $f : x \to a(x - h)^2 + k$,

$a < 0$, $h > 1$

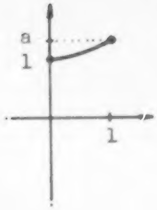


(vii) $f : x \to a(x - h)^2 + k$,

$a > 0$, $h < \frac{1}{2}$

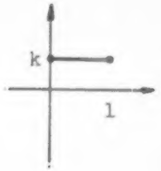


(viii) $f : x \to a^x$, $a > 1$

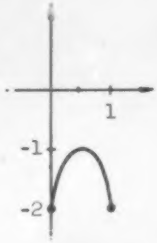


(c) $c = \frac{1}{2}$, f is differentiable at c.

(i) $f : x \to k$

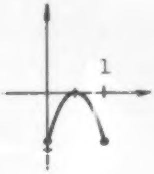


(ii) $f : x \to -\csc(\frac{\pi}{2} x + - \frac{\pi}{6})$

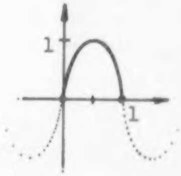


(iii) $f : x \to k(x - \frac{1}{2})^2$,

$k < 0$, n is an even positive integer.

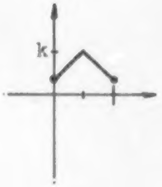


(iv) $f : x \to \sin \pi x$

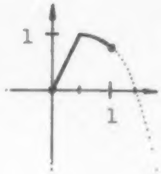


(d) $c = \frac{1}{2}$ and the graph has a corner at c.

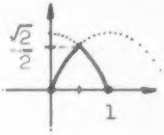
(i) $f : x \to -|m(x - \frac{1}{2})| + b, \ m \neq 0$



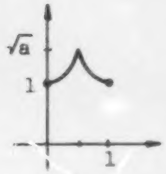
(ii) $f : x \to \begin{cases} 2x, & \text{if } 0 \leq x < \frac{1}{2} \\ -(x - \frac{1}{2})^2 + 1, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$



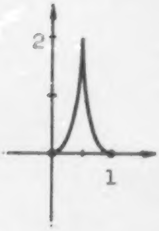
(iii) $f : x \to \begin{cases} \sin \frac{\pi}{2} x, & \text{if } 0 \leq x < \frac{1}{2} \\ \cos \frac{\pi}{2} x, & \text{if } \frac{1}{2} \leq x < \frac{1}{2} \end{cases}$



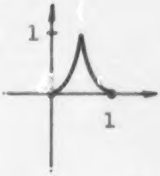
(iv) $f : x \to \begin{cases} a^x, & \text{if } 0 \le x < \frac{1}{2} \\ a^{1-x}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$



(v) $f : x \to \begin{cases} 8x^2, & \text{if } 0 \le x < \frac{1}{2} \\ 8(x-1)^2, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

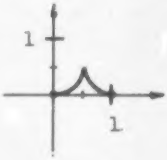


(vi) $f : x \to \begin{cases} \tan \frac{\pi}{2} x, & \text{if } 0 \le x \le \frac{1}{2} \\ \cot \frac{\pi}{2} x, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

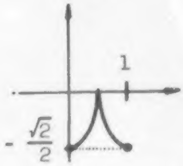


(e) $c = \frac{1}{2}$ and the graph has a vertical tangent at c.

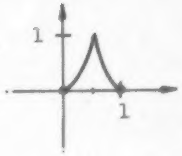
(i) $f : x \to \begin{cases} \frac{1}{2} - \sqrt{\frac{1}{4} - x^2}, & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2} - \sqrt{\frac{1}{4} - (x-1)^2}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$



(ii) $f : x \to \begin{cases} -\sqrt{\frac{1}{2} - x}, & \text{if } 0 \le x < \frac{1}{2} \\ -\sqrt{x - \frac{1}{2}}, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$



(iii) $f : x \to [1 - |2x - 1|^{1/2}]^2$



13. f is a polynomial function of degree 2, on $0 \leq x \leq 1$ such that there are two distinct points c_1 and c_2 on the interval where f is maximum.

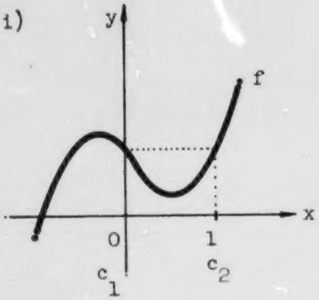
(a) c_1 and c_2 must be end points.

(b) There can be no third maximum point.

(c) If f is maximum at the end points, since it is differentiabe on the interval, then the minimum is at c such that $f'(c) = 0$ by Theorem 8-2c. Since the derivative of a second degree polynomial is a first degree polynomial, there is but one solution to the equation $f'(c) = 0$. Thus c is unique.

14. f is a polynomial of degree three.

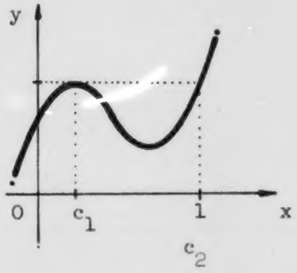
(a) (i)



$c_1 = 0$

$c_2 = 1$

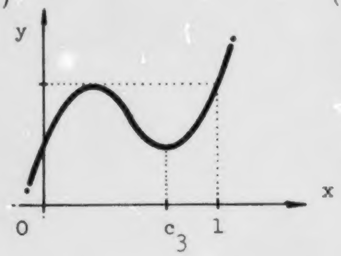
(ii)



$c_2 = 1$, c_1 is on an interior point such that $f'(c_1) = 0$.

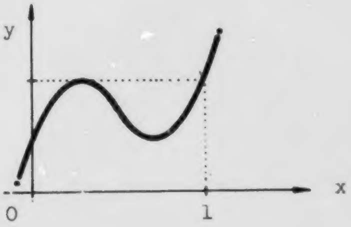
(b) There is no third maximum point on the interval $0 \leq x \leq 1$.

(c) (i)



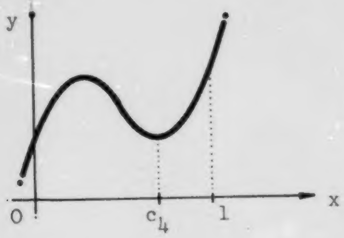
c_3 is an interior point such that $f'(c_3) = 0$

(ii)



$c_3 = 0$

(d) f with exactly two minimums, having two maximums.



$c_3 = 0$, c_4 is an interior point such that $f'(c_4) = 0$.

15. f is a polynomial function

(a) If f is of degree 2 then f'' is a constant. Thus, f is strictly convex if $f'' = k > 0$ and strictly concave if $f'' = k < 0$.

(b) If f is of degree 3 then f'' is a linear function, $f''(x) = mx + b$, $m \neq 0$. When $x > -\frac{b}{m}$, f is strictly convex for $m > 0$ and strictly concave for $m < 0$. When $x < -\frac{b}{m}$, f is strictly concave for $m > 0$ and strictly convex for $m < 0$.

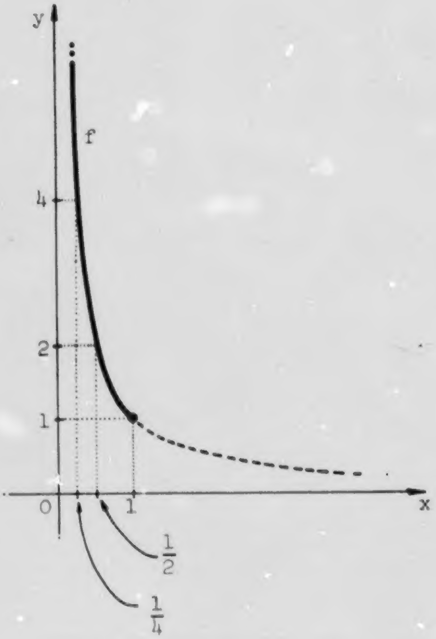
(c) If f is of degree 4 then f'' is of degree 2. In this case since there are two zeros to f'' there are three intervals if both zeros are real. Suppose these zeros are x_1 and x_2, such that $x_1 < x_2$. Then the interval $-\infty \leq x \leq x_1$ and $x_2 \leq x \leq \infty$ are both strictly convex or strictly concave; the interval $x_1 \leq x \leq x_2$ will be the opposite.

If the two zeros of f'' are complex then f is strictly convex or strictly concave over the entire domain.

If the two zeros of f'' are alike, that is if $|f''(x)| = (x - a)^2$ then f is strictly convex or strictly concave over the entire domain.

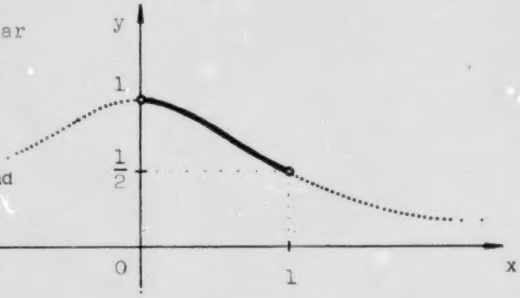
Solutions Exercises 8-2b

1. (a) $f : x \to \frac{1}{x}$, $0 < x \leq 1$



We can suppose that f is bounded above by say $f(\epsilon)$ for some $\epsilon > 0$ no matter how small. But f is also defined for $\frac{\epsilon}{2}$. Then there is another number $f(\frac{\epsilon}{2}) = 2f(\epsilon)$. We are forced to admit that $f(\epsilon)$ is not an upper bound. Thus given any candidate for an upper bound we can exhibit another candidate larger than the first. Therefore, no upper bound exists on the interval $0 < x \leq 1$.

(b) $f : x \to \frac{1}{1 + x^2}$ on $0 < x < 1$.



By a line of reasoning similar to part (a) we suppose that $f(\epsilon)$ is a maximum for some very small $\epsilon > 0$. But for the value $\frac{\epsilon}{2}$ between 0 and ϵ, $f(\frac{\epsilon}{2}) > f(\epsilon)$. Therefore, no matter what ϵ on the interval we select there is another which can be found to produce a larger maximum.

Suppose that $f(1 - \epsilon)$ is a minimum. But $1 - \frac{\epsilon}{2}$ is on the interval $1 - \epsilon < 1 - \frac{\epsilon}{2} < 1$ and $f(1 - \frac{\epsilon}{2}) < f(1 - \epsilon)$. Thus there is no value of x on the interval $0 < x < 1$ such that $f(x)$ is a maximum or a minimum.

2. If f is continuous and strictly increasing on $0 \leq x \leq 1$ and $f(0) < d < f(1)$, then there exists a unique c, $0 < c < 1$, such that $f(c) = d$.

   By the Intermediate Value Theorem there is at least one c. Assume that there exists at least two values, c_1 and c_2, such that $f(c_1) = d = f(c_2)$ and $c_1 < c_2$.

   By the definition of strictly increasing if $c_1 < c_2$ then $f(c_1) < f(c_2)$. Thus we arrive at a contradiction.

   Our assumption is false. Therefore, there is only one value of c which satisfies the stated conditions.

3. If f is continuous then by Theorem 8-2b there is a maximum and a minimum value of f on the interval $a \leq x \leq b$. That is, there exists a c and a d, $a \leq c \leq b$ and $a \leq d \leq b$ such that $f(c) \leq f(x) \leq f(d)$ for all x on $a \leq x \leq b$.

   By the Intermediate Value Theorem for every D; $f(c) \leq D \leq f(d)$ there is an e such that $f(e) = D$ on the closed interval with c and d as end points.

   If $c < d$ then $c \leq e \leq d$.
   If $c > d$ then $d \leq e \leq c$.

   Thus each image D on the interval $f(c) \leq d \leq f(d)$ has a pre-image e on the closed interval with end points c and d. Since f is continuous there are no gaps in the set of pre-images and every image has a pre-image.

4. (a) If $f'(x)$ is continuous and $f'(x) > 0$ on $a \leq x \leq b$ then f is increasing on the interval by Theorem 8-2e. Thus for $a \leq x \leq b$, then $f(a) \leq f(x) \leq f(b)$.

   (b) If f' is continuous and positive then f is continuous and strictly increasing. Thus by Exercise 4 for every image d on $f(a) \leq d \leq f(b)$, there is a unique pre-image c, on $a \leq c \leq b$.

   Let g be the function that maps every image into its pre-image. Thus $g(d) = g(f(c)) = c$. Or, in general, $g(f(x)) = x$.

5. A rigorous argument is not anticipated.

   By Theorem 8-2c, f has at least one minimum. If f has two minima say c_1 and c_2, then there is a point between them, say d, such that $f(d) > f(c_1)$ and $f(d) > f(c_2)$. But this contradicts the concept of strictly convex, since $f(d)$ must fall below the chord $\overline{c_1c_2}$ if f is strictly convex.

The minimum will occur at a if $f'(x) \geq 0$ on the entire interval.

The minimum will occur at b if $f'(x) \leq 0$ on the entire interval.

Otherwise, $f'(x)$ should be negative for part of the interval and positive for the remainder, then the minimum will occur at a unique interior point of the interval.

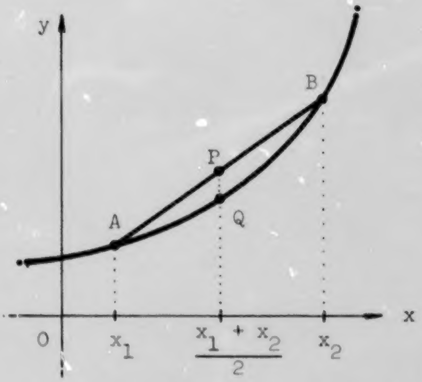
6. (a) If f is convex then

$$P\left(\frac{x_1 + x_2}{2}, \frac{f(x_1) + f(x_2)}{2}\right)$$

and

$$Q\left(\frac{x_1 + x_2}{2}, f\left(\frac{x_1 + x_2}{2}\right)\right)$$

are the points in question.



The intuitive concept of convexity requires that all values of $f(x)$ on the interval $x_1 \leq x \leq x_2$ fall either on or below the closed $\overline{AB}$.

That is
$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2} .$$

7. (a) $\frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0 = f'(c)$ for c in $a < c < b$.

(b) If f is continuous in $a \leq x \leq b$, differentiable in $a < x < b$ and $f(a) = f(b) = 0$, then there is a number c such that $f'(c) = 0$ and $a < c < b$.

If $f : x \to 0$ the statement is obvious. Suppose f is not the constant function. By Theorem 8-2b there is at least one maximum and one minimum in the interval. If either occurs at an endpoint then the other occurs in the interior of the interval. That is, at least one extremum c occurs in the interval $a < c < b$. Since f is differentiable then $f'(c) = 0$ by Theorem 8-2c.

8. Theorem 8-2e. Suppose that f is differentiable in the interval $a \leq x \leq b$ and that f' is continuous and nonnegative in the interval. Then f is increasing in the interval.

Let us examine the difference quotient

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$$

when $a \leq x_1 < x_2 \leq b$.

By the Mean Value Theorem

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0)$$

for since x_0 in the interval $x_1 \leq x_0 \leq x_2$. Since $x_1 < x_2$ then $x_2 - x_1 > 0$. By our hypothesis $f'(x_0) \geq 0$ for all x_0 in the interval $a \leq x_0 \leq b$.

Then $f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1) \geq 0$, $f(x_2) > f(x_1)$ and f is an increasing function.

9. If $f'(x) = 0$ in $a \leq x \leq b$, then f is constant. Consider $f(x_1)$ and $f(x_2)$ such that $a \leq x_1 < x_2 \leq b$. By the Mean Value Theorem for some x_0 in $x_1 \leq x_0 \leq x_2$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0)$$

and

$$f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1).$$

But $f'(x_0) = 0$ and $x_2 - x_1 \neq 0$. Thus $f(x_2) - f(x_1) = 0$ and f is the constant function.

Solutions Exercises 8-3

1. (a) $y = x^{1/3} - 3x^{-2/5}$

$y' = \frac{1}{3} x^{-2/3} + \frac{6}{5} x^{-7/5}$

(b) $y = x^2 + 2 \sin x$

$y' = 2x + 2 \cos x$

(c) $y = (3x^2 + 1)(x^4 + 1)$

$= 3x^6 + x^4 + 3x^2 + 1$

$y' = 18x^5 + 4x^3 + 6x$

(d) $y = (1 - 2x)\left(\frac{1}{x^2} + \frac{1}{x}\right)$

$= (1 - 2x)(x^{-2} + x^{-1})$

$= x^{-2} - x^{-1} - 2$

$y' = -2x^{-3} + x^{-2}$

(e) $y = e^x + e^{2x} + \cos x$

$y' = e^x + 2e^{2x} - \sin x$

(f) $y = \sqrt{x} - 3e^{-x}$

$y' = \frac{1}{2} x^{-1/2} + 3e^{-x}$

(g) $y = x + \log_e x^2 - 2 \log_e x$

$= x + 2 \log_e x - 2 \log_e x$

$= x$

$y' = 1$

(h) $y = x^e + e^x$

$y' = ex^{e-1} + e^x$

$= e(x^{e-1} + e^{x-1})$

2. $f : x \to \sqrt{x} + \frac{1}{x}$, for $0 < x \leq 1$

$f(\frac{1}{2}) = \frac{\sqrt{2}}{2} + 2$

$u : x \to \sqrt{x}$

$u(\frac{1}{2}) = \frac{\sqrt{2}}{2}$

$v : x \to \frac{1}{x}$

$v(\frac{1}{2}) = 2$

$f' : x \to \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

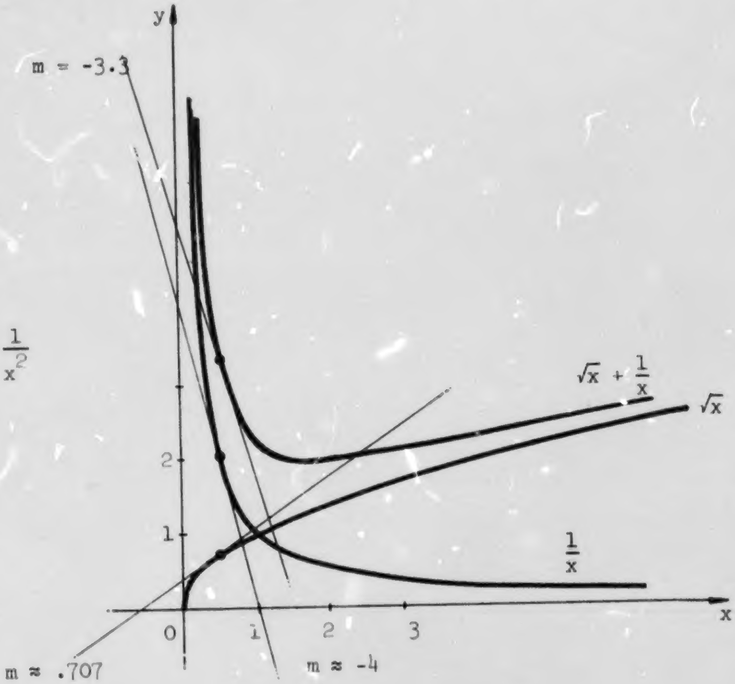
$f'(\frac{1}{2}) = \frac{\sqrt{2}}{2} - 4$

$u' : x \to \frac{1}{2\sqrt{x}}$

$u'(\frac{1}{2}) = \frac{\sqrt{2}}{2}$

$v' : x \to -\frac{1}{x^2}$

$v'(\frac{1}{2}) = -4$



Tangent lines at $x = \frac{1}{2}$

to f : $y - (\frac{\sqrt{2}}{2} + 2) = (\frac{\sqrt{2}}{2} - 4)(x - \frac{1}{2})$

to u : $y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x - \frac{1}{2})$

to v : $y - 2 = -4(x - \frac{1}{2})$

The equation of the tangent line to f is the linear combination of the tangent lines to u and to v.

3. (a) $y = \sin x - \sqrt{3} \cos x$

$y' = \cos x + \sqrt{3} \sin x$

The tangent line is horizontal when $y' = 0$.

$$0 = \cos x + \sqrt{3} \sin x$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = -\frac{\pi}{6} + n\pi, \ n = 0, \pm 1, \pm 2, \ldots$$

(b) $y = 2^x - 2x$

$y' = 2^x \log_e 2 - 2$

This will be perpendicular to $y = 3x + 2$ if $y' = -\frac{1}{3}$.

$$-\frac{1}{3} = 2^x \log_e 2 - 2$$

$$2^x = \frac{5}{3 \log_e 2}$$

$$x \log_e 2 = \log_e\left(\frac{5}{3 \log_e 2}\right)$$

$$x = \frac{\log_e 5 - \log_e 3 - \log_e(\log_e 2)}{\log_e 2}$$

$$x = \frac{1.61 - 1.79 - \log_e(0.69)}{0.69}$$

$$\approx \frac{1.61 - 1.99 - (-0.37)}{.69}$$

$$\approx -.014$$

(c) If the tangent lines of $y = 5f(x)$ and $y = 7f(x)$ are parallel at $x = a$ then $5f'(a) = 7f'(a)$ or $2f'(a) = 0$ which implies that $f'(a) = 0$. Thus $y = 5f(x)$ and $y = 7f(x)$ both have horizontal tangent lines at $x = a$.

(d) If u and v are differentiable then $f'(x) = u'(x) + 3v'(x)$ and $g'(x) = u'(x) - 11v'(x)$. Since $f'(a) = g'(a)$ then $3v'(a) = -11v'(a)$ or $14v'(a) = 0$ and $v'(a) = 0$ which means that the tangent to v at $(a,v(a))$ is horizontal.

4. If a and b are constants then

$$\begin{aligned} D(av + bu) &= D(av) + D(bu) && \text{by (1)} \\ &= a\,D(v) + b\,D(u) && \text{by (2).} \end{aligned}$$

5. (a) $f : x \to x - \cos x, \quad 0 \le x \le 2\pi$

$f' : x \to 1 + \sin x$

$f'' : x \to \cos x$

(i) f is increasing in $0 \le x \le 2\pi$

(ii) f is convex in $0 \le x \le \frac{\pi}{2}$

and in $\frac{3\pi}{2} < x \le 2\pi$

f is concave in $\frac{\pi}{2} < x \le \frac{3\pi}{2}$

(iii) No asymptotes.

(b) $f : x \to e^x - 2x, \quad 0 \le x \le 1$

$f' : x \to e^x - 2$

$f'' : x \to e^x$

(i) f is increasing when $e^x \ge 2$

$0.69 < x \le 1$

f is decreasing when $0 \le x \le 0.69$

(ii) f is everywhere convex.

(iii) No asymptotes in $0 \le x \le 1$.

There is the asymptote $y = -2x$ for f when $x \to -\infty$.

(c) $f : t \to t^2 + \frac{3}{t}, \quad 0 < t$

$f' : t \to 2t - \frac{3}{t^2}$

$f'' : t \to 2 + \frac{6}{t^3}$

(i) f is increasing when $\sqrt[3]{\frac{3}{2}} \le t$

f is decreasing in $0 < t < \sqrt[3]{\frac{3}{2}}$

(ii) f is convex for $0 < t$

(iii) A vertical asymptote as $t \to 0$.

(d) $f : x \to x^2 - \sqrt{2x}, \quad 0 \leq x \leq 2$

$f' : x \to 2x - \dfrac{\sqrt{2}}{2\sqrt{x}}$

$f'' : x \to 2 + \dfrac{\sqrt{2}}{4x^{3/2}}$

(i) f is increasing if $\frac{1}{2} < x \leq 2$

f is decreasing if $0 \leq x \leq \frac{1}{2}$

(ii) f is convex in $0 \leq x \leq 2$

(iii) No asymptotes.

6. (a) Given that $F(x) = \int_x^b f$. By the Area Theorem let $G' = f$ where G is an antiderivative of f.

$$\int_x^b f = G(b) - G(x) = F(x)$$

Then

$$\begin{aligned} F'(x) &= G'(b) - G'(x) \\ &= 0 - f(x) \\ &= -f(x) \end{aligned}$$

(b) If $F(x) = \int_x^0 e^{-t^2}\,dt$ then $F'(x) = -e^{-x^2}$ by part (a).

7. The motion of a particle is defined as

$$s(t) = 2 \cos t + t^2$$

the velocity as $v(t) = s'(t) = -2 \sin t + 2t$ and the acceleration as $a(t) = v'(t) = s''(t) = -2 \cos t + 2$. Since $|-2 \cos t| \leq 2$ then $0 \leq -2 \cos t + 2 \leq 4$ and the acceleration is always nonnegative.

8. Consider the polynomial function $P : x \to a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$.

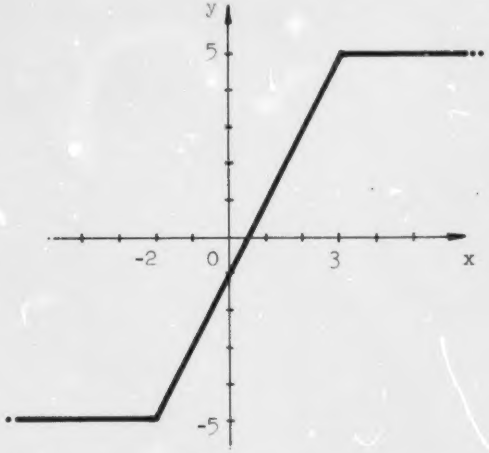
$$\begin{aligned} P' : x \to\ & D(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) \\ &= D(a_0) + D(a_1x) + D(a_2x^2) + \ldots + D(a_nx^n) \quad \text{by (1)} \\ &= a_0\, D(x^0) + a_1\, D(x) + a_2\, D(x^2) + \ldots\ a_n\, D(x^n) \quad \text{by (2).} \end{aligned}$$

Finally using $Dx^n = nx^{n-1}$.

$$P' : x \to a_0 \cdot 0x^{-1} + a_1 \cdot 1 \cdot x^0 + a_2 \cdot 2x^1 + \ldots a_n \cdot n \cdot x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1}$$

9. (a)



(b) $$|x + 2| = \begin{cases} x + 2, & \text{if } x \geq -2 \\ -(x + 2), & \text{if } x < -2 \end{cases}$$

$$|3 - x| = \begin{cases} 3 - x, & \text{if } x \leq 3 \\ x - 3, & \text{if } x > 3 \end{cases}$$

$$g(x) = -(x + 2) - (3 - x) = -5, \quad \text{if } x < -2$$
$$g(x) = (x + 2) - (3 - x) = 2x - 1, \quad \text{if } -2 \leq x \leq 3$$
$$g(x) = (x + 2) - (-3 + x) = 5, \quad \text{if } x > 3$$

(c) f' is not defined at $x = -2$ and at $x = 3$.

10. (a) $1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + x^2, \quad 0 \leq x \leq 1$

Let $f(x) = e^x - (1 + x + \frac{x^2}{2})$

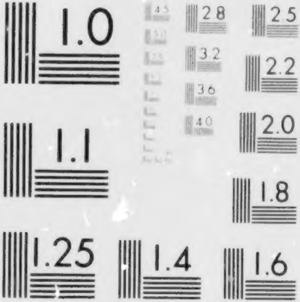
$$f'(x) = e^x - 1 - x$$

The minimum f occurs when $f'(x) = 0$. Since $f'(0) = 0$ we have found at least one minimum.

Let $$g(x) = 1 + x + x^2 - e^x$$

$$g'(x) = 1 + 2x - e^x.$$

Again a minimum occurs when $x = 0$.


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

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(b) (In 6-4-(4) one form of this problem was first introduced.)

Let $f(x) = v(x) - u(x)$

Then $f'(x) = v'(x) - u'(x)$ by (1).

Since $v'(x) \geq u'(x)$ it follows that

$$f'(x) \geq 0.$$

Thus, by Theorem 8-2e f is an increasing function. When $a \leq x$ then

$$f(a) \leq f(x)$$

and $v(a) - u(a) \leq v(x) - u(x)$.

Since $u(a) \leq v(a)$,

$$0 \leq v(a) - u(a)$$

implies that $0 \leq v(x) - u(x)$

and $u(x) \leq v(x)$ for $a \leq x$.

(c) From part (b) since $u'(a) \leq v'(a)$ and $D(u'(x)) \leq D(v'(x))$ for $a \leq x$ then $u'(x) \leq v'(x)$ for $a \leq x$.

But we now have exactly the conditions of part (b), thus $u(x) \leq v(x)$ for $a \leq x$.

11. (a) If $y = u$ and $y = v$ are solutions

$$y'' - 3y' + 6y = 0$$

then $u'' - 3u' + 6u = 0$

and $v'' - 3v' + 6v = 0$.

Substitute $y = 3u + 8v$

$$y' = 3u' + 8v'$$

and $y'' = 3u'' + 8v''$.

$$(3u'' + 8v'') - 3(3u' + 8v') + 6(3u + 8v)$$

$$= 3(u'' - 3u' + 6u) + 8(v'' - 3v' + 6v)$$

$$3(0) + 8(0) = 0$$

Thus, $y = 3u + 8v$ is also a solution.

(b) If $y = e^x + e^{-x}$ If $y = e^x - e^{-x}$

$$y' = e^x - e^{-x} \qquad y' = e^x + e^{-x}$$

$$y'' = e^x + e^{-x}, \qquad y'' = e^x - e^{-x}.$$

In each case $y'' = y$.

If $y = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x})$ for α, β constants.

$$y' = \alpha(e^x - e^{-x}) + \beta(e^x + e^{-x})$$

$$y'' = \alpha(e^x + e^{-x}) + \beta(e^x - e^{-x}).$$

Again $y'' = y$.

12. $u(x) = v(x) + ax + b$, for a, b constants.

(a) $u'(x) = v'(x) + a$ and $u'(x) - v'(x) = a$.

(b) Since $D(a) = 0$ then $u''(x) = v''(x)$.

(c) If $u'' = v''$ then $u' = v' + a$ by the Constant Difference Theorem. Since $u' = v' + a$ then $u = v + ax + b$ by the Constant Difference Theorem.

Then $u - v = ax + b$, a linear function.

13. If u and v are continuous at $x = a$ then $u(a)$ and $v(a)$ are defined and $\lim\limits_{x \to a} u(x) = u(a)$ and $\lim\limits_{x \to a} v(x) = v(a)$.

Examine

$$f = 2u - 3v$$

$$f(a) = 2u(a) - 3v(a)$$

is defined and

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a} (2u(x) - 3v(x)) \\ &= \lim_{x \to a} 2u(x) - \lim_{x \to a} 3v(x) \\ &= 2 \lim_{x \to a} u(x) - 3 \lim_{x \to a} v(x) \\ &= 2u(a) - 3v(a) \\ &= f(a).\end{aligned}$$

Thus f is continuous.

14. The fact that f is differentiable at $x = a$ does not insure that both u and v are also differentiable at $x = a$.

Here are three examples of functions of the form $f = u + v$ such that f is differentiable at a but u and v are not necessarily differentiable at a.

(i) $f = |x - a| + (-|x - a|)$

(ii) $f = u + v$, where $u = \begin{cases} 0, & x < a \\ 2x, & a \leq x \end{cases}$

and $v = \begin{cases} 2x, & x < a \\ 0, & a \leq x \end{cases}$

(iii) $f = u + v$, where $u = \begin{cases} x + 1, & \text{if } x \text{ is rational} \\ 1\ , & \text{if } x \text{ is irrational.} \end{cases}$

and $v = \begin{cases} -1\ , & \text{if } x \text{ is rational} \\ x - 1, & \text{if } x \text{ is irrational} \end{cases}$

Solutions Exercises 8-4

1. (a) $a_1 = a^2, \quad m_1 = 2a$

$a_2 = a^3, \quad m_2 = 3a^2$

(b) $y_1 \cdot y_2 = (a^2 + 2a(x - a))(a^3 + 3a^2(x - a))$

$$= a^5 + 5a^4(x - a) + 6a^3(x - a)^2$$

If we omit the last term then the expression is $a^5 + 5a^4(x - a)$. If $u \cdot v = f : x \to x^5$ then $f(a) = a^5$ and $f'(a) = m = 5a^4$. Thus the tangent line is

$$y = a^5 + 5a^4(x - a),$$

which is the desired result.

2. (a) $Dx(2x - 3) = 4x - 3$

(b) $D(4x - 2)(4 - 2x) = -16x + 20$

(c) $D(x^2 + x + 1)(x^2 - x + 1) = 4x^3 + 2x$

(d) $D\sqrt{x}\,(ax + b)^3 = \dfrac{(ax + b)^3}{2\sqrt{x}} + 3a\sqrt{x}\,(ax + b)^2$

(e) $D(\frac{1}{x} \cdot \sqrt{x}) = -\frac{1}{2}x^{-3/2}$

(f) $D(\frac{1}{x}(5x + 2)) = -\dfrac{2}{x^2}$

(g) $D(xe^x) = e^x + x\,e^x$

$= e^x(1 + x)$

(h) $Dx^{7/2} = \frac{7}{2}x^{5/2}$

(i) $D(3x^4 + \frac{1}{\sqrt{x}}) = 12x^3 - \frac{1}{2}x^{-3/2}$

(j) $D(3x^2(x^2 - 5)) = 12x^3 - 30x$

(k) $D(\sqrt{x}\cos 2x) = -2\sqrt{x}\sin 2x + \dfrac{1}{2\sqrt{x}}\cos 2x$

(ℓ) $D(e^{3x}\sin(x + 1)) = e^{3x}\cos(x + 1) + 3e^{3x}\sin(x + 1)$

$= e^{3x}(\cos(x + 1) + 3\sin(x + 1))$

(m) $D(x^2 \log_e x) = x + 2x \log_e x = x(1 + 2 \log_e x)$

(n) $D((x - 1)^{1/2} e^{-x}) = -(x - 1)^{1/2} e^{-x} + \frac{1}{2(x - 1)^{1/2}} e^{-x}$

(o) $D(x\int_0^x e^{-t^2}\, dt) = xe^{-x^2} + \int_0^x e^{-t^2}\, dt$

(p) $D(e^x \int_1^x \frac{\sin t}{t}\, dt) = e^x \frac{\sin x}{x} + e^x \int_1^x \frac{\sin^t}{t}\, dt$

(q) $D(xe^x \sin x) = xe^x \cos x + xe^x \sin x + e^x \sin x$

$= e^x(x \cos x + (x + 1)\sin x)$

(r) $D((\log_e x)(4x^2 + 2x)(\cos 2x))$

$= -2 \log_e x(4x^2 + 2x)\sin 2x + (\log_e x)(\cos 2x)(8x + 2)$

$+ (4x + 2)(\cos 2x)$

(s) $D(2 \sin x \cos x) = -2 \sin^2 x + 2 \cos^2 x$

$= 2(\cos^2 x - \sin^2 x)$

$= 2 \cos 2x$

This was not unexpected since

$$2 \sin x \cos x = \sin 2x$$

and $$D(\sin 2x) = 2 \cos 2x.$$

(t) $D(xe^x \log_e (2x + 1)\sin x) = xe^x \log_e (2x + 1)\cos x + \frac{2xe^x}{2x + 1} \sin x$

$+ xe^x \log_e (2x + 1)\sin x$

$+ e^x \log_e (2x + 1)\sin x$

$= e^x(x \log(2x + 1)\cos x + \frac{2x}{2x + 1} \sin x$

$+ (x + 1) \log_e (2x + 1) \sin x)$

(u) $D(x^2 2^x) = x^2(\log_e 2)2^x + 2x2^x$

$= x^2 2^x (\frac{2}{x} + \log_e 2)$

(v) $D(x \log_2 (3x + 1)) = \frac{3x}{(\log_e 2)(3x + 1)} + \log_2 (3x + 1)$

$= \frac{1}{\log_e 2} (\frac{3x}{3x + 1} + \log_e (3x + 1))$

(w) $D(x^e e^x) = x^e e^x + ee^x x^{e-1}$

$= x^e e^x (1 + \frac{e}{x})$

3. (a) $D(3x^2 + 5x - 1)^2 = D(3x^2 + 5x - 1)(3x^2 + 5x - 1)$

$= (3x^2 + 5x - 1)(6x + 5) + (6x + 5)(3x^2 + 2x - 1)$

$= 2(6x + 5)(3x^2 + 5x - 1)$

(b) $D(3 - 5x)^3 = D(3 - 5x)(3 - 5x)(3 - 5x)$

$= (3 - 5x)(3 - 5x)(-5) + (3 - 5x)(-5)(3 - 5x)$

$+ (-5)(3 - 5x)(3 - 5x)$

$= -15(3 - 5x)^2$

(c) $D(3 - 5x)^4 = D(3 - 5x)^3(3 - 5x)$

$= (3 - 5x)^3(-5) - 15(3 - 5x)^2(3 - 5x)$

$= -20(3 - 5x)^3$

(d) $D(x(\sqrt{x} - 1)^2) = D(x(\sqrt{x} - 1)(\sqrt{x} - 1))$

$= \frac{x(\sqrt{x} - 1)}{2\sqrt{x}} + \frac{x(\sqrt{x} - 1)}{2\sqrt{x}} + (\sqrt{x} - 1)^2$

$= (\sqrt{x} - 1)(\frac{\sqrt{x}}{2} + \frac{\sqrt{x}}{2} + \sqrt{x} - 1)$

$= (\sqrt{x} - 1)(2\sqrt{x} - 1)$

or $2x - 3\sqrt{x} + 1$

(e) $D(x + \frac{1}{x})^2 = D((x + \frac{1}{x})(x + \frac{1}{x}))$

$= (x + \frac{1}{x})(1 - \frac{1}{x^2}) + (1 - \frac{1}{x^2})(x + \frac{1}{x})$

$= \frac{2}{x}(x^2 + 1) \frac{1}{x^2}(x^2 - 1)$

$= \frac{2}{x^3}(x^4 - 1)$

or $2x - \frac{2}{x^3}$

(f) $D\left(\frac{x^{3/2}}{3} - \frac{x^{1/2}}{2} + x^{-1/2}\right) = \frac{x^{1/2}}{2} - \frac{x^{-1/2}}{4} - \frac{x^{-3/2}}{2}$.

(g) $D\left(4\sqrt{x^3} - 2\sqrt{x} + \frac{1}{\sqrt{x}}\right) = 6\sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

(h) $D(e^x \sin(1 - 2x)) = -2e^x \cos(1 - 2x) + e^x \sin(1 - 2x)$

$= e^x(-2 \cos(1 - 2x) + \sin(1 - 2x))$

(i) $D(\sqrt{x} \log_e x) = \frac{\sqrt{x}}{x} + \frac{1}{2\sqrt{x}} \log_e x$

$= \frac{1}{\sqrt{x}}(1 + \frac{1}{2} \log_e x)$

or $\frac{1}{\sqrt{x}}(1 + \log_e \sqrt{x})$

(j) $D(x^\pi \pi^x) = x^\pi(\log_e \pi)\pi^x + \pi x^{\pi-1}\pi^x$

$= x^\pi \pi^x(\frac{\pi}{x} + \log_e \pi)$

(k) $D(x^2 \cos x) = -x^2 \sin x + 2x \cos x$

$= x(2 \cos x - x \sin x)$

(ℓ) $D(\sin x \log_e x) = \frac{\sin x}{x} + \cos x \log_e x$

(m) $D\left(\frac{\log_e x}{x}\right) = \frac{1}{x} \cdot \frac{1}{x} + \left(-\frac{1}{x^2}\right)\log_e x$

$= \frac{1}{x^2}(1 - \log_e x)$

4. (a) If $f(x) = [u(x)]^2$

then $f(x) = u(x) \cdot u(x)$

and $f'(x) = u(x)u'(x) + u'(x)u(x)$

$= 2u(x)u'(x)$

(b) If $f(x) = [u(x)]^3$

then $f(x) = u(x)[u(x)]^2$

and $D[u(x)]^3 = u(x)D[u(x)]^2 + u'(x)[u(x)]^2$

$= u(x)[2u(x)(u'(x))] + u'(x)[u(x)]^2$

$= 3[u(x)]^2u'(x)$

(c) $D[u(x)]^4 = D[u(x) \cdot (u(x))^3]$

$= u(x)[3(u(x))^2u'(x)] + u'(x)[u(x)]^3$

$= 4[u(x)]^3u'(x)$

(d) $D[u(x)]^n = n[u(x)]^{n-1}u'(x)$

5. (a) $y = \sin^2 x$

$y' = 2(\sin x)\cos x$

(b) $y = \cos^3(4x)$

$y' = -12 \cos^2(4x)\sin 4x$

(c) $y = (\log_e x)^2$

$y' = \frac{2}{x} \log_e x$

(d) $y = (e^x)^4$

$y' = 4(e^x)^3 \cdot e^x$

$= 4(e^x)^4$

(e) $y = (x^2 + 1)^2$

$y' = 4x(x^2 + 1)$

(f) $y = \sin^3(2x - 1)$

$y' = 6 \sin^2(2x - 1)\cos(2x - 1)$

(g) $y = \left(\int_1^x \sin t^2 \, dt\right)^4$

$y' = 4\left(\int_1^x \sin t^2 \, dt\right)^3 \sin x^2$

6. (a) $y = x^2(x^2 + 1)^2$

$y' = x^2[2(x^2 + 1)2x] + 2x(x^2 + 1)^2$

$= 2x(x^2 + 1)(2x^2 + x^2 + 1)$

$= 2x(x^2 + 1)(3x^2 + 1)$

or $6x^5 + 8x^3 + 2x$

(b) $y = (x + 1)^3(x^2 - x + 1)$

$$y' = (x + 1)^3(2x - 1) + 3(x + 1)^2(x^2 - x + 1)$$

$$= (x + 1)^2((x + 1)(2x - 1) + 3(x^2 - x + 1))$$

$$= (x + 1)^2(5x^2 - 2x + 2)$$

(c) $y = (ax^2 + bx + c)(dx^2 + ex + f)$

$$y' = (ax^2 + bx + c)(2dx + e) + (2ax + b)(dx^2 + ex + f)$$

$$= 4adx^3 + 3(ae + bd)x^2 + 2(cd + af + be)x + (ce + bf)$$

(d) $y = (\cos^2 x)\sin 2x$

$$y' = 2 \cos^2 x \cos 2x - 2 \cos x \sin x \sin 2x$$

$$= 2 \cos x(\cos x \cos 2x - \sin x \sin 2x)$$

$$= 2 \cos x \cos 3x$$

(e) $y = e^x \sin^2(ax + b)$

$$y' = 2ae^x \sin(ax + b)\cos(ax + b) + e^x \sin^2(ax + b)$$

$$= e^x \sin(ax + b)(2a \cos(ax + b) + \sin(ax + b))$$

(f) $y = (x\int_0^x e^{t^2} dt)^2 = x^2(\int_0^x e^{t^2} dt)^2$

$$y' = 2x^2e^{x^2} (\int_0^x e^{t^2} dt)^1 + 2x (\int_0^x e^{t^2} dt)^2$$

$$= 2x\int_0^x e^{t^2} dt[xe^{x^2} + \int_0^x e^{t^2} dt]$$

(g) $y = x^3[\log_e(x + 1)]^3$

$$y' = \frac{3x^3}{x + 1}[\log_e(x + 1)]^2 + 3x^2[\log_e(x + 1)]^3$$

$$= 3x^2[\log_e(x + 1)]^2[\frac{x}{x + 1} + \log_e(x + 1)]$$

7. (a) $y = x \log_e x \ , \quad x > 0$

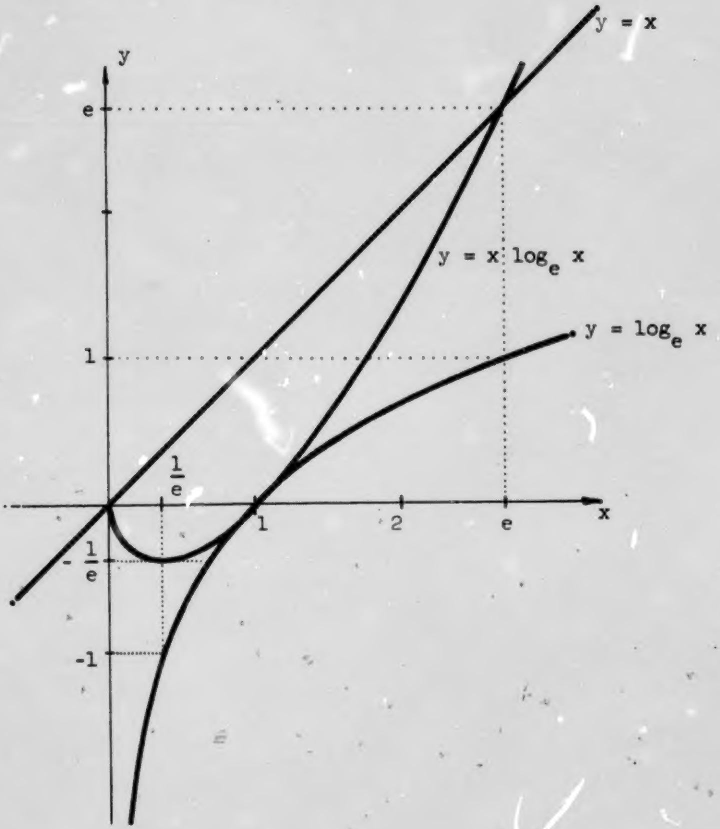
$y' = 1 + \log_e x$

$y'' = \frac{1}{x}$

The graph increases if $x \geq \frac{1}{e}$

The graph decreases if $0 < x < \frac{1}{e}$

The graph is convex if $0 < x$



(b) $y = \frac{1}{x} \log_e x, \quad x > 0$

$$y' = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \log_e x$$

$$= \frac{1}{x^2}(1 - \log_e x)$$

y is increasing when $\frac{1}{x^2}(1 - \log_e x) > 0$ or when $x \leq e$.

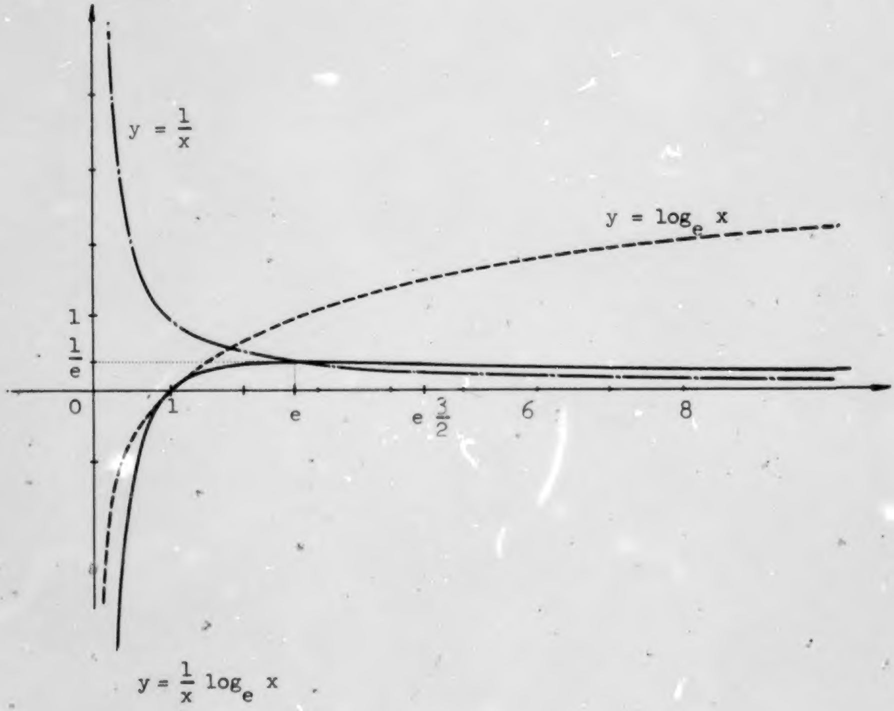
y is decreasing when $x > e$.

$$y'' = \frac{1}{x^2}(-\frac{1}{x}) + (-\frac{2}{x^3})(1 - \log_e x)$$

$$= \frac{1}{x^3}(2 \log_e x - 3)$$

y is convex if $\frac{1}{x^2}(2 \log_e x - 3) \geq 0$ or when $x \geq e^{3/2} \approx 4.48$.

y is concave if $0 < x < e^{3/2}$



(c) $y = \sin 3x, \quad 0 \leq x \leq 2\pi$

$y' = 3 \sin^2 x \cos x$

$y'' = 3(-\sin^3 x + 2 \sin x \cos^2 x)$

We see that y' depends solely upon $\cos x$ for its being positive or negative.

Thus y is increasing when $\cos x \geq 0$, that is when $0 \leq x < \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq x \leq 2\pi$ and y is decreasing when $\frac{\pi}{2} \leq x < \frac{3\pi}{2}$. Analyzing y'' is more involved.

$$y'' = 3 \sin^3 x(2 \cot^2 x - 1).$$

We see that $y'' \geq 0$ if

(i) both $\sin^3 x \geq 0$

and $2 \cot^2 x - 1 \geq 0$ or

(ii) both $\sin^3 x < 0$

and $2 \cot^2 x - 1 < 0$.

Case (i): $\sin^3 x \geq 0$ when $0 \leq x < \pi$

$2 \cot^2 x - 1 \geq 0$

$\cot^2 x \geq \frac{1}{2}$

$$|\cot x| \geq .707, \quad \cot .955 \approx .707$$

Thus

$$0 \leq x < .955 \text{ or } \pi - .955 \leq x < \pi + .955 \text{ or } \pi - .955 \leq x \leq 2\pi.$$

Combining both conditions of case (i) $y'' \geq 0$ when

$$0 \leq x < .955 \text{ or } \pi - .955 \leq x < \pi.$$

Case (ii): $\sin^3 x \leq 0$ when $\pi \leq x \leq 2\pi$

$2 \cot^2 x - 1 \leq 0$ when $.955 \leq x < \pi - .955$
or $\pi + .955 \leq x < 2\pi - .955$

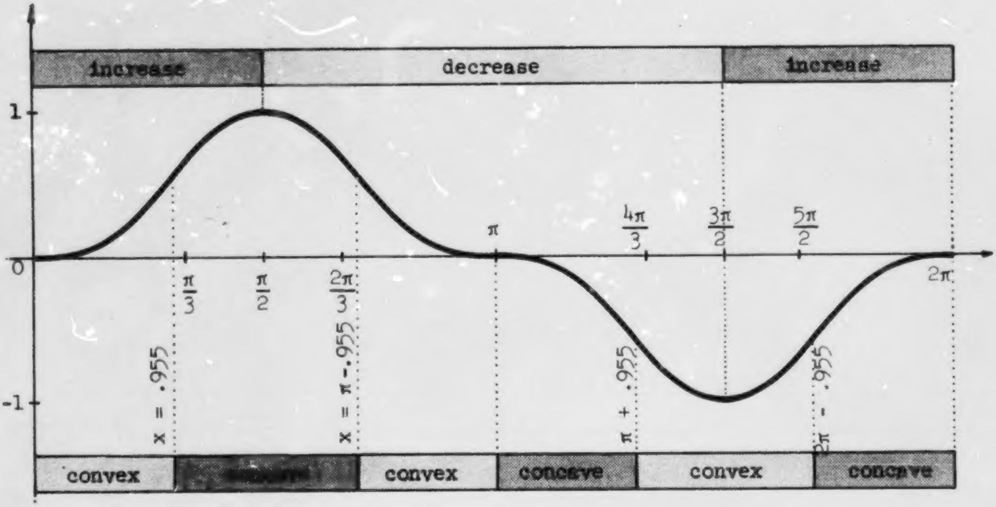
Combining both conditions of case (ii) $y'' \geq 0$ when

$$\pi + .955 \leq x < 2\pi - .955.$$

Thus y is convex in the following domain $0 \leq x < .955$, $\pi - .955 \leq x < \pi$, or $\pi + .955 \leq x < 2\pi - .955$.

We can assume that y is concave in the complement of the domain for which $y'' \geq 0$, namely

$$.955 \leq x < \pi - .955, \quad \pi \leq x < \pi + .955, \quad \text{or} \quad 2\pi - .955 \leq x \leq 2\pi.$$



(d) $y = x^2 \log_e x, \quad x > 0$

$$y' = x + 2x \log_e x$$

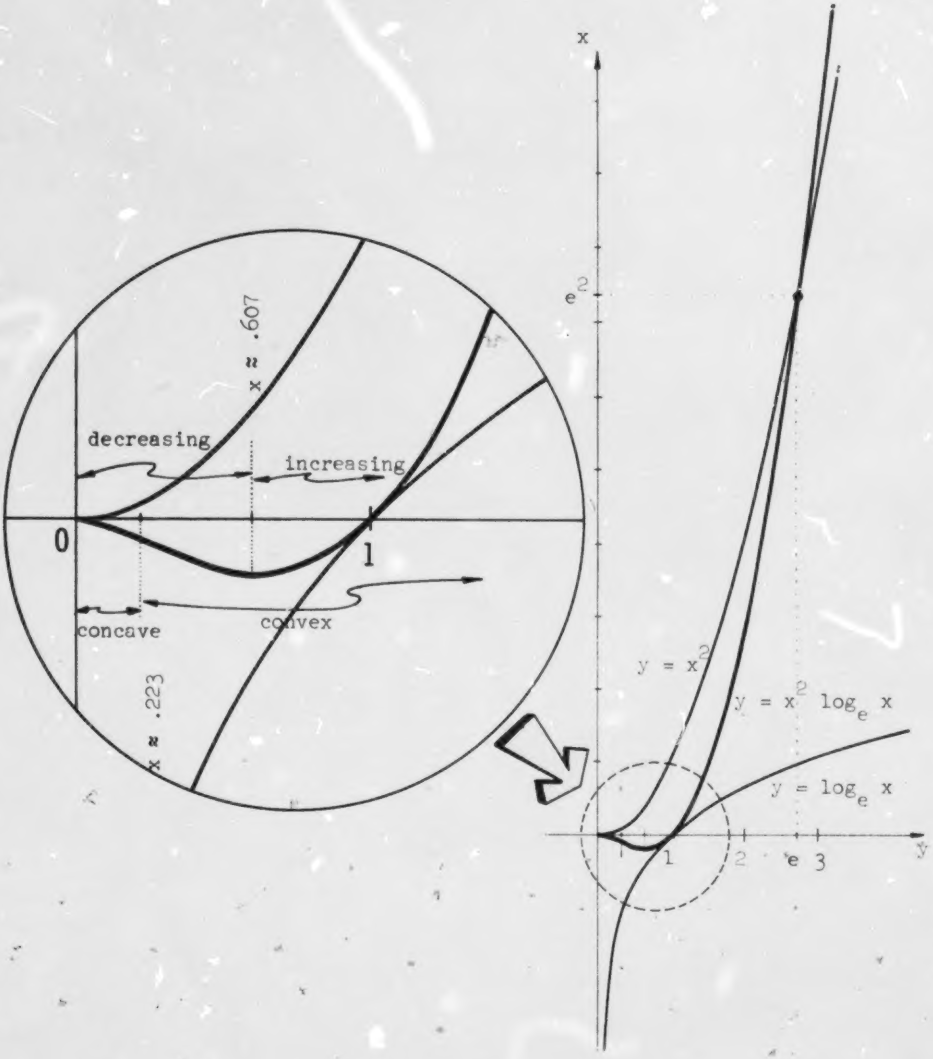
$$= x(1 + 2 \log_e x)$$

$$y'' = 2 + (1 + 2 \log_e x)$$

$$= 3 + 2 \log_e x$$

Since $x > 0$ then $y' \geq 0$ when $1 + 2 \log_e x \geq 0$ or when $x \geq e^{-1/2} \approx .607$. Thus y increases when $x \geq e^{-1/2}$ and decreases if $0 < x < e^{-1/2}$.

The function is convex when $3 + 2 \log_e x \geq 0$ or when $x \geq e^{-3/2} \approx .223$ and concave for $x \leq e^{-3/2}$.



8. (a) $x \to \sqrt{x}\, e^x, \quad x > 0$

$$D(\sqrt{x}\, e^x) = \sqrt{x}\, e^x + \frac{e^x}{2\sqrt{x}}$$

$$= \sqrt{x}\, e^x\left(1 + \frac{1}{2x}\right)$$

Since $x > 0$ then $\sqrt{x} > 0$, $e^x > 0$ and $\left(1 + \frac{1}{2x}\right) > 0$.

Thus $D(\sqrt{x}\, e^x) > 0$ and the function is increasing.

(b) $x \to \frac{e^x}{x}$, $x \geq 1$

$$D\left(\frac{e^x}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$= \frac{e^x}{x^2}(x - 1)$$

If $x \geq 1$ then $e^x > 0$, $x^2 > 0$ and $(x - 1) \geq 0$.

Thus $D\left(\frac{e^x}{x}\right) \geq 0$ and the function is increasing.

(c) $x \to \frac{e^x}{x^\alpha}$, $x \geq \alpha > 0$

$$D\left(\frac{e^x}{x^\alpha}\right) = \frac{e^x}{x^\alpha} - \alpha \frac{e^x}{x^{\alpha+1}}$$

$$= \frac{e^x}{x^{\alpha+1}}(x - \alpha)$$

If $x \geq \alpha > 0$ then $e^x > 0$, $x^{\alpha+1} > 0$ and $(x - \alpha) \geq 0$.

Thus $D\left(\frac{e^x}{x^\alpha}\right) \geq 0$ and the function is increasing.

(d) $x \to x \sin x$, $0 \leq x \leq \frac{\pi}{2}$

$D(x \sin x) = x \cos x + \sin x$

If $0 \leq x \leq \frac{\pi}{2}$ then $x > 0$, $\sin x \geq 0$ and $\cos x \geq 0$.

Thus $D(x \sin x) \geq 0$ and the function is increasing.

9. If $f(x) = (x - a)^2 g(x)$, $g(a) \neq 0$ and g is differentiable then

$$f'(x) = (x - a)^2 g'(x) + 2(x - a)g(x).$$

Since $(x - a)$ is a factor of each term then $f'(a) = 0$.

10. Let $D : x \rightarrow a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ be a polynomial function with the zero a of multiplicity at least 3. Then by the Factor Theorem $P(x) = (x - a)^3Q(x)$ where $Q(x)$ is a polynomial of degree $(n - 3)$.

$$P'(x) = (x - a)^3Q'(x) + 3(x - a)^2Q(x)$$

$$P''(x) = (x - a)^3Q''(x) + 3(x - a)^2Q'(x)$$
$$+ 3(x - a)^2Q'(x) + 6(x - a)Q(x)$$

The expression for $P''(x)$ has a factor of $(x - a)$ in each term. Thus, $P''(a) = 0$ whenever a is a zero of the polynomial of multiplicity greater than two.

The converse is not true. A counter example is called for. Assume $a \neq 0$.

Suppose that $P''(x) = x - a$, and $P''(a) = 0$.

Further $P'(x) = \frac{x^2}{2} - ax$,

and $P(x) = \frac{x^3}{6} - \frac{ax^2}{2}$.

But $P(a) \neq 0$.

11. (a) If $y = e^{ax} \cos bx$

then $y' = -be^{ax} \sin bx + ae^{ax} \cos bx$

and $y'' = -b^2 e^{ax} \cos bx - ab\, e^{ax} \sin bx$

$-ab\, e^{ax} \sin bx + a^2 e^{ax} \cos bx$

$= e^{ax}(a^2 \cos bx - 2ab \sin bx - b^2 \cos bx)$.

Substituting into $y'' - 2ay + (a^2 + b^2)y$ we get

$$e^{ax}(a^2 \cos bx - 2ab \sin bx - b^2 \cos bx) - e^{ax}(-2ab \sin bx + 2a^2 \cos bx)$$
$$+ e^{ax}(a^2 \cos bx + b^2 \cos bx).$$

Simplifying yields

$$e^{ax}((a^2 - 2a^2 + a^2 - b^2 + b^2)\cos bx + (-2ab + 2ab)\sin bx)$$

or

$$e^{ax}(0 \cdot \cos bx + 0 \cdot \sin bx)$$

which is clearly zero.

(b) If $\quad y = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

then $\quad y' = e^x(2x + 2) + e^x(x^2 + 2x)$

$$= e^x(x^2 + 4x + 2)$$

$$y'' = e^x(2x + 4) + e^x(x^2 + 4x + 2)$$

$$= e^x(x^2 + 6x + 6)$$

and $\quad y''' = e^x(2x + 6) + e^x(x^2 + 6x + 6)$

$$= e^x(x^2 + 8x + 12)$$

Substituting into $y''' - 3y'' + 3y' - y$ gives

$$e^x(x^2 + 8x + 12) - e^x(3x^2 + 18x + 18) + e^x(3x^2 + 12x + 6) - e^x(x^2 + 2x).$$

Simplifying yields

$$e^x((1 - 3 + 3 - 1)x^2 + (8 - 18 + 12 - 2)x + (12 - 18 + 6))$$

or $\quad e^x(0 \cdot x^2 + 0 \cdot x + 0)$

which is clearly zero.

12. (a) $(uv)' = uv' + vu'$

$$(uv)'' = uv'' + u'v' + u'v' + u''v$$

$$= uv'' + 2u'v' + u''v.$$

(b) $f : x \to x^2 \cos x$

Let $u = x^2$ and $v = \cos x$. Then $u' = 2x$, $u'' = 2$, $v' = -\sin x$ and $v'' = -\cos x$.

Thus from (a)

$$f''(x) = -x^2 \cos x - 4x \sin x + 2 \cos x.$$

(c) $(uv)''' = uv''' + u'v'' + 2u'v'' + 2u''v' + u'''v + u'' + v'$

$$= uv''' + 3u'v'' + 3u''v' + u'''v.$$

(d) $$(uv)^n = uv^{(n)} + nu'v^{(n-1)} + \frac{n \cdot (n - 1)}{2} u''v^{(n-2)} + \ldots$$

$$+ \frac{n!}{m!(n - m)!} u^m v^{(n-m)} + \ldots$$

$$+ nu^{(n-1)}v' + u^{(n)}v \qquad \text{for } m \leq n.$$

The coefficients are the coefficients of the binomial expansion.

Solutions Exercises 8-5

1. (a) $x \to \sqrt{1 - x^2}$

Let $f(u) = \sqrt{u}$

and $u(x) = 1 - x^2.$

Or let $f(u) = \sqrt{1 - u}$

and $u(x) = x^2.$

Then $f(u(x)) = \sqrt{1 - x^2}.$

(b) $x \to e^{x^2}$

Let $f(u) = e^u$

and $u(x) = x^2.$

Then $f(u(x)) = e^{x^2}.$

(c) $x \to \cos(x^3 - 3x)$

Let $f(u) = \cos u$

and $u(x) = x^3 - 3x.$

Then $f(u(x)) = \cos(x^3 - 3x).$

(d) $x \to \frac{1}{1 + x^2}$

Let $f(u) = \frac{1}{u}$

and $u(x) = 1 + x^2.$

Or let $f(u) = \frac{1}{1 + u}$

and $u(x) = x^2.$

Then $f(u(x)) = \frac{1}{1 + x^2}.$

(e) $x \to \log_e \sqrt{x^2 + 1}$

Let $f(u) = \log_e u$

and $u(x) = \sqrt{x^2 + 1}.$

Or let $f(u) = \log_e \sqrt{u + 1}$

and $u(x) = x^2$.

Or let $f(u) = \frac{1}{2} \log_e u$

and $u(x) = x^2 + 1$.

Or let $f(u) = \frac{1}{2} \log (u + 1)$

and $u(x) = x^2$.

Then $f(u(x)) = \frac{1}{2} \log_e (x^2 + 1)$

$= \log_e \sqrt{x^2 + 1}$

(f) $x \to (2 - 3x^2)^{100}$

Let $f(u) = (2 - u)^{100}$

and $u(x) = 3x^2$.

Or let $f(u) = u^{100}$

and $u(x) = 2 - 3x^2$.

Then $f(u(x)) = (2 - 3x^2)^{100}$.

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

Let $f(u) = u^{-1/2}$

and $u(x) = 2x^2 - 2x + 1$.

Then $f(u(x)) = (2x^2 - 2x + 1)^{-1/2}$.

(h) $x \to \log_e(\sin x)^2$

Let $f(u) = \log_e u$

and $u(x) = \sin^2 x$.

Or let $f(u) = \log_e u^2$

and $u(x) = \sin x$.

Or let $f(u) = 2 \log_e u$

and $u(x) = \sin x$

Then $f(u(x)) = 2 \log_e \sin x$

$= \log_e(\sin x)^2$.

(i) $x \rightarrow e^{\cos^2 x}$

Let $f(u) = e^u$

and $u(x) = \cos^2 x.$

Or let $f(u) = e^{u^2}$

and $u(x) = \cos x.$

Then $f(u(x)) = e^{(\cos x)^2}.$

(j) $x \rightarrow 3e^{2 \sin x}$

Let $f(u) = 3u$

and $u(x) = e^{2 \sin x}.$

Or let $f(u) = 3e^u$

and $u(x) = 2 \sin x.$

Or let $f(u) = 3e^{2u}$

and $u(x) = \sin x.$

Then $f(u(x)) = 3e^{2 \sin x}.$

(k) $x \rightarrow 2^{(x+1)^2}$

Let $f(u) = 2^u$

and $u(x) = (x + 1)^2.$

Or let $f(u) = 2^{u^2}$

and $u(x) = x + 1.$

Then $f(u(x)) = 2^{(x+1)^2}.$

2. (a) $x \rightarrow \log_e |8x^2 + 5x + 2|$

Let $f(u) = \log_e u$

$u(v) = |v|$

and $v(x) = 8x^2 + 5x + 2.$

Then $f(u(v(x))) = \log_e |8x^2 + 5x + 2|.$

(b) $x \to \sqrt{1 + \cos x}$

Let $f(u) = \sqrt{u}$

$u(v) = 1 + v$

and $v(x) = \cos x.$

Then $f(u(v(x))) = \sqrt{1 + \cos x}.$

(c) $x \to \cos(\sin(\cos x))$

Let $f(u) = \cos u$

$u(v) = \sin v$

and $v(x) = \cos x.$

Then $f(u(v(x))) = \cos(\sin(\cos x)).$

(d) $x \to (x + 1)^{3/5}$

Let $f(u) = (u)^{1/5}$

$u(v) = v^3$

and $v(x) = x + 1.$

Or let $f(u) = u^3$

$u(v) = v^{1/5}$

and $v(x) = x + 1.$

Then $f(u(v(x))) = (x + 1)^{3/5}.$

(e) $x \to \sqrt{1 - (\log_e x)^2}$

Let $f(u) = \sqrt{u}$

$u(v) = 1 - v$

$v(q) = q^2$

and $q(x) = \log_e x.$

Then $f(u(v(q(x)))) = \sqrt{1 - (\log_e x)^2}.$

(f) $x \rightarrow \dfrac{1}{1 + e^{2x}}$

Let
$$f(u) = \frac{1}{u}$$
$$u(v) = 1 + v$$
$$v(q) = e^q$$
$$q(x) = 2x$$

Then
$$f(u(v(q(x)))) = \frac{1}{1 + e^{2x}}$$

3. If $x \rightarrow |x|$ then let $f(u) = \sqrt{u}$ and $u(x) = x^2$. Thus $f(u(x)) = \sqrt{x^2}$ which is another way of saying $|x|$.

4. (a) Let $f(u) = au + b$ and $u(x) = px + q$ be linear functions.

Then
$$\begin{aligned} f(u(x)) &= a(px + q) + b \\ &= apx + aq + b \end{aligned}$$

which is a linear function.

(b) Let
$$f(u) = au^2 + bu + c$$
and
$$u(x) = rx^2 + sx + t$$
be quadratic functions.

Then
$$\begin{aligned} f(u(x)) &= a(rx^2 + sx + t)^2 + b(rx^2 + sx + t) + c \\ &= a(r^2x^4 + s^2x^2 + t^2 + 2rsx^3 + 2rtx^2 + 2stx) \\ &\qquad + b(rx^2 + sx + t) + c \\ &= ar^2x^4 + 2rsx^3 + (as^2 + wrt + br)x^2 \\ &\qquad + (2st + bs)x + (at^2 + bt + c) \end{aligned}$$

which is a fourth degree polynomial.

(c) The composite of two polynomial functions is again a polynomial function. The degree of the composite polynomial function will be equal to the product of the degrees of the two contributing functions.

5. (a) If $u : x \rightarrow x$ and $f : x \rightarrow u(u(x))$ then
$$\begin{aligned} f(3) &= u(u(3)) \\ &= u(3) \\ &= 3. \end{aligned}$$

(b) If $u : x \to \frac{1}{x}$ then $f : x \to u(u(x))$. Thus $f(x) = u(u(x)) = u(\frac{1}{x}) = x$ and $f(x) = x$.

6. (a) If $u : x \to x^a$ and $v : x \to x^b$ then

$$u(v(x)) = u(x^b)$$
$$= (x^b)^a$$
$$= x^{ab}$$

and $$v(u(x)) = v(x^a)$$
$$= (x^a)^b$$
$$= x^{ab}.$$

Thus $$u(v(x)) = v(u(x)).$$

(b) If $u : x \to \cos x$ and $v : x \to \sin x$ then

$$u(v(x)) = \cos(\sin x)$$

and $$v(u(x)) = \sin(\cos x).$$

A counter example will establish the fact that $u(v(x)) \neq v(u(x))$.

Let $$x = \frac{\pi}{2},$$

then $$u(v(x)) = \cos(\sin \frac{\pi}{2})$$
$$= \cos 1$$
$$\approx .5403.$$

But $$v(u(x)) = \sin(\cos \frac{\pi}{2})$$
$$= \sin 0$$
$$= 0$$

$$0 \neq .5403$$

Therefore $$u(v(x)) \neq v(u(x)).$$

(c) Let $u : x \to a^x$ and $v : x \to b^x$

$$u(v(x)) = a^{(b^x)}$$

and $$v(u(x)) = b^{(a^x)}.$$

In general $u(v(x)) \neq v(u(x))$ unless $a = b$. The counter example which gives the quickest results is to let $x = 0$.

(d) Let $u : x \to e^x$ and $v : x \to \log_e x$ then

$$u(v(x)) = e^{\log_e x} = x$$

and $$v(u(x)) = \log_e e^x = x.$$

In this case $$u(v(x)) = v(u(x)).$$

This is no surprise since $\log_e x$ was introduced as the inverse function of e^x.

7. (a) $x \to \int_{-2}^{x^2} t^{2/3}\,dt$

Let $$f(u) = \int_{-2}^{u} t^{2/3}\,dt$$

and $$u(x) = x^2,$$

then $$f(u(x)) = \int_{-2}^{x^2} t^{2/3}\,dt.$$

(b) $x \to \int_{\sin x}^{1} e^t\,dt$

Let $$f(u) = \int_{u}^{1} e^t\,dt$$

and $$u(x) = \sin x.$$

Then $$f(u(x)) = \int_{\sin x}^{1} e^t\,dt.$$

(c) $x \to \int_{0}^{x^2} e^{-t^2}\,dt$

Let $$f(u) = \int_{0}^{u} e^{-t^2}\,dt$$

and $$u(x) = x^2.$$

Then $$f(u(x)) = \int_{0}^{x^2} e^{-t^2}\,dt.$$

8. $x \to \sqrt{1 - (\log_e x)^2}$

First, $\log_e x$ is only defined when $x > 0$. Secondly, $\sqrt{1 - (\log_e x)^2}$ is only defined for real numbers when

$$1 - (\log_e x)^2 \geq 0.$$

Thus

$$-(\log_e x)^2 \geq -1$$

$$(\log_e x)^2 \leq 1$$

$$|\log_e x| \leq 1$$

$$-1 \leq \log_e x \leq 1$$

$$\frac{1}{e} \leq x \leq e$$

The domain of the function is the interval $\frac{1}{e} \leq x \leq e$.

Solutions Exercises 8-6

1. (a) $x \to \sqrt{1 - x^2}$

Let $f(u) = u^{1/2}$

and $u(x) = 1 - x^2$.

Then $f'(x) = f'(u) \cdot u'(x) = \dfrac{1}{2\sqrt{1 - x^2}} \cdot -2x$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

(b) $x \to e^{x^2}$

$$f(u) = e^u$$

$$u(x) = x^2$$

$$f'(x) = e^{x^2} \cdot 2x$$

$$= 2x\, e^{x^2}$$

(c) $x \to \cos\,(x^3 - 3x)$

$$f(u) = \cos u$$

$$u(x) = x^3 - 3x$$

$$f'(x) = -\sin(x^3 - 3x) \cdot (3x^2 - 3)$$

$$= (3 - 3x^2)\sin(x^3 - 3x)$$

(d) $x \to \dfrac{1}{1 + x^2}$

$$f(u) = \frac{1}{u}$$

$$u(x) = 1 + x^2$$

$$f'(x) = -\frac{1}{(1 + x^2)^2} \cdot (2x)$$

$$= \frac{-2x}{(1 + x^2)^2}$$

(e) $x \to \log_e \sqrt{x^2 + 1}$

$$f(u) = \log u$$

$$u(v) = \sqrt{v}$$

$$v(x) = x^2 + 1$$

$$f'(x) = \frac{1}{\sqrt{x^2 + 1}} \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x$$

$$= \frac{x}{x^2 + 1}$$

(f) $x \to (2 - 3x^2)^{100}$

$$f(u) = u^{100}$$

$$u(x) = 2 - 3x^2$$

$$f'(x) = 100(2 - 3x^2)^{99} \cdot (-6x)$$

$$= -600x(2 - 3x^2)^{99}$$

(g) $x \to (2x^2 - 2x + 1)^{-1/2}$

$$f(u) = u^{-1/2}$$

$$u(x) = 2x^2 - 2x + 1$$

$$f'(x) = -\frac{1}{2}(2x^2 - 2x + 1)^{-3/2} \cdot (4x - 2)$$

$$= (1 - 2x)(2x^2 - 2x + 1)^{-3/2}$$

(h) $x \to \log_e (\sin x)^2$

$$f(u) = \log u$$

$$u(v) = v^2$$

$$v(x) = \sin x$$

$$f'(x) = \frac{1}{\sin^2 x} \cdot 2 \sin x \cdot \cos x$$

$$= 2 \cot x$$

(i) $x \to e^{\cos^2 x}$

$$f(u) = e^u$$

$$u(v) = v^2$$

$$v(x) = \cos x$$

$$f'(x) = e^{\cos^2 x} \cdot 2 \cos x \cdot (-\sin x)$$

$$= -2e^{\cos^2 x}(\cos x \sin x)$$

or $$-e^{\cos^2 x} (\sin 2x)$$

(j) $x \to 3e^{2 \sin x}$

$$f(u) = 3e^{2u}$$

$$u(x) = \sin x$$

$$f'(x) = 6e^{2 \sin x} (\cos x)$$

(k) $x \to 2^{(x+1)^2}$

$$f(u) = 2^u$$

$$u(v) = v^2$$

$$v(x) = x + 1$$

$$f'(x) = (\log_e 2) \cdot (2^{(x+1)^2}) \cdot 2(x + 1) \cdot (1)$$

$$= 2(x + 1)(\log_e 2)(2^{(x+1)^2})$$

2. (a) $x \to \sqrt{1 + \cos x}$

$$f(u) = \sqrt{u}$$

$$u(x) = 1 + \cos x$$

$$f'(x) = \frac{1}{2\sqrt{1 + \cos x}} \cdot (-\sin x)$$

$$= \frac{-\sin x}{2\sqrt{1 + \cos x}}$$

(b) $x \to \sqrt{1 - (\log_e x)^2}$

$$f(u) = \sqrt{u}$$

$$u(v) = 1 - v^2$$

$$v(x) = \log_e x$$

$$f'(x) = \frac{1}{2\sqrt{1 - (\log_e x)^2}} \cdot (-2 \log_e x) \cdot \frac{1}{x}$$

$$= \frac{-\log_e x}{x\sqrt{1 - (\log_e x)^2}}$$

(c) $x \to \dfrac{1}{1 + e^{2x}}$

$$f(u) = \frac{1}{u}$$

$$u(x) = 1 + e^{2x}$$

$$f'(x) = \frac{-1}{(1 + e^{2x})^2} \cdot 2e^{2x}$$

$$= \frac{-2e^{2x}}{(1 + e^{2x})^2}$$

(d) $x \to \cos(\sin(\cos x))$

$$f(u) = \cos u$$

$$u(v) = \sin v$$

$$v(x) = \cos x$$

$$f'(x) = [-\sin(\sin(\cos x))](\cos(\cos x))(-\sin x)$$

3. (a) $x \to (x^2 + 1)^{1/2} + (x^2 + 1)^{-1/2}$

$$f'(x) = \frac{1}{2(x^2 + 1)^{1/2}} \cdot 2x + \frac{-1}{2(x^2 + 1)^{3/2}} \cdot 2x$$

$$= \frac{x(x^2 + 1) - x}{(x^2 + 1)^{3/2}}$$

$$= \frac{x^3}{(x^2 + 1)^{3/2}}$$

(b) $x \to (x^2 - a^2)^{1/2} \cdot (x^2 + a^2)^{-1/2}$

$$f'(x) = (x^2 - a^2)^{1/2}(-\tfrac{1}{2})(x^2 + a^2)^{-3/2}(2x) + (x^2 + a^2)^{-1/2}(\tfrac{1}{2})$$

$$(x^2 - a^2)^{-1/2}(2x)$$

$$= (x^2 - a^2)^{-1/2}(x^2 + a^2)^{-3/2}(-x(x^2 - a^2) + x(x^2 + a^2))$$

$$= 2a^2\, x(x^2 - a^2)^{-1/2}(x^2 + a^2)^{-3/2}$$

or $$\frac{2a^2 x}{(x^2 - a^2)^{1/2}(x^2 + a^2)^{3/2}}$$

(c) $x \to x(2x^2 + 2x + 1)^{-1/2}$

$$f'(x) = x(-\tfrac{1}{2})(2x^2 + 2x + 1)^{-3/2}(4x + 2) + (2x^2 + 2x + 1)^{-1/2}$$

$$= (2x^2 + 2x + 1)^{-3/2}(-(2x^2 + x) + (2x^2 + 2x + 1))$$

$$= (x + 1)(2x^2 + 2x + 1)^{-3/2}$$

(d) $x \to x^2 \sqrt{\sin x}$

$$f'(x) = x^2 \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x + \sqrt{\sin x}\,(2x)$$

$$= \frac{x}{2} \sqrt{\sin x}\,(x \cot x + 4)$$

(e) $x \to \sin^2 (e^x)$

$$f'(x) = 2 \sin e^x \cdot \cos e^x \cdot e^x$$

$$= 2e^x \sin e^x \cos e^x$$

or $$e^x \sin 2e^x$$

(f) $x \to e^{x \sin x}$

$$f'(x) = e^{x \sin x}(x \cos x + \sin x)$$

(g) $x \to \log_e(\sqrt{x} \cos x)$

$$f'(x) = \frac{1}{\sqrt{x} \cos x} \cdot (\sqrt{x}(-\sin x) + \frac{1}{2\sqrt{x}} \cdot \cos x)$$

$$= -\tan x + \frac{1}{2x}$$

(h) $x \to e^{\log_e x + \cos x}$

$$f'(x) = (\frac{1}{x} - \sin x)e^{\log_e x + \cos x}$$

(i) $x \to \sin x \cos x \log_e \sqrt{x}$

$$f'(x) = \sin x \cos x(\frac{1}{2x}) + \sin x(-\sin x)\log_e \sqrt{x} + \cos x \cos x \log_e \sqrt{x}$$

$$= \frac{1}{2x} \sin x \cos x + \log_e \sqrt{x}\,(\cos^2 x - \sin^2 x)$$

or $$\frac{1}{4x} (\sin 2x) + \log_e \sqrt{x}\,(\cos 2x)$$

Alternate method

$$x \to \frac{1}{2} \sin 2x \log_e \sqrt{x}$$

$$f'(x) = \frac{1}{2} \sin 2x\left[\frac{1}{2x}\right] + \left(\frac{2}{2}\right)\log_e \sqrt{x}\ (\cos 2x)$$

$$= \frac{1}{4x} (\sin 2x) + (\log_e \sqrt{x})(\cos 2x)$$

(j) $x \to \cos^2(\log_e x) + \sin^2(\log_e x)$

$$f'(x) = 0$$

4. (a) $f(x) = \int_a^{g(x)} h(t)\,dt$

Let $\quad u = g(x)$

$$f(u) = \int_a^u h(t)\,dt$$

then $\quad f'(u) = h(u)$

and $\quad f'(x) = f'(u) \cdot g'(x)$

$$= h(g(x))g'(x)$$

(b) $F(x) = \int_{x^2}^{b} f = -\int_b^{x^2} f$

Let $\quad x^2 = u.$

Then $\quad F'(u) = -f(u)$

and $\quad u'(x) = 2x.$

Thus $\quad F'(x) = -f(x^2)2x$

$$= -2x\ f(x^2)$$

(c) $f(x) = \int_{-\pi}^{x^2} \sin t \, dt = -\cos t \Big|_{-\pi}^{x^2}$

$$= (-\cos x^2) - (-\cos(-\pi))$$

$$= -\cos x^2 - 1$$

$f'(x) = -(-\sin x^2)(2x)$

$= 2x \sin x^2$

If we allow $x^2 = u$ then the first result is the same as

$$f'(u) \cdot u'(x) = \sin(x^2)(2x).$$

5. (a) $x \to \int_{-2}^{x^2} t^{2/3} \, dt$

$$f'(x) = (x^2)^{2/3} \cdot 2x$$

$$= 2x^{7/3}$$

(b) $x \to \int_{\sin x}^{1} e^t \, dt$

$$f'(x) = -e^{\sin x} \cdot (\cos x)$$

(c) $x \to \int_{0}^{x^2} e^{-t^2} \, dt$

$$f'(x) = e^{-x^4} \cdot 2x$$

$$= 2xe^{-x^4}$$

6. (a) $f : x \to x^x, \quad x > 0$

$f : x \to e^{x \log_e x}$

$f'(x) = e^{x \log_e x} \cdot (x \cdot \frac{1}{x} + \log_e x)$

$= x^x(1 + \log_e x)$

(b) Minimum value of f occurs when $f' = 0$.

Since $\quad x > 0$ then $x^x > 0$.

Thus $\quad f' = 0$ when $1 + \log_e x = 0$

$$1 + \log_e x = 0$$
$$\log_e x = -1$$
$$x = e^{-1}$$

The minimum value of f is

$$f(e^{-1}) = (e^{-1})^{e^{-1}}$$
$$= \left(\frac{1}{e}\right)^{1/e} \approx (.3679)^{(.3679)} \approx .711$$

(c) $f''(x) = x^x\left(\frac{1}{x}\right) + x^x(1 + \log_e x)^2$

$$= x^x\left(\frac{1}{x} + (1 + \log_e x)^2\right)$$

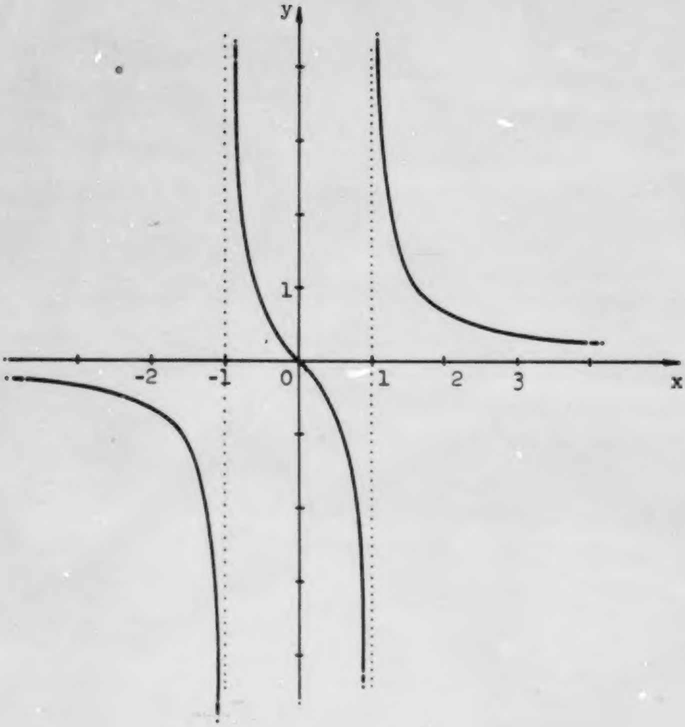
By inspection since $x > 0$ then $f'' > 0$ and f is convex.

7. $f : x \to \dfrac{x}{x^2 - 1}$

$$f'(x) = x \cdot \frac{-1}{(x^2 - 1)^2} \cdot 2x + \frac{1}{x^2 - 1}$$
$$= \frac{-2x^2 + (x^2 - 1)}{(x^2 - 1)^2}$$
$$= \frac{-x^2 - 1}{(x^2 - 1)^2} = \frac{-(x^2 + 1)}{(x^2 - 1)^2}$$
$$x^2 + 1 > 0 \text{ and } (x^2 - 1)^2 > 0.$$

Thus $f'(x) < 0$ for all $x \neq \pm 1$.

$$f''(x) = -(x^2 + 1) \cdot \frac{(-2)}{(x^2 - 1)^3} \cdot 2x + \frac{-(2x)}{(x^2 - 1)^2}$$
$$= \frac{4x(x^2 + 1) - 2x(x^2 - 1)}{(x^2 - 1)^3}$$
$$= \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$$



$x^2 + 3 > 0$ for all values of x.

$(x^2 - 1)^3 < 0$ when $-1 < x < 1$

$(x^2 - 1)^3 \geq 0$ when $1 \leq |x|$

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
$2x$	-	-	+	+
$x^2 + 3$	+	+	+	+
$(x^2 - 1)^2$	+	-	-	+
$f''(x)$	-	+	-	+

f is concave if $x < -1$

f is convex if $-1 < x < 0$

f is concave if $0 \leq x < 1$

f is convex if $1 < x$.

(b) $f : x \to e^{1/x}$

$$f'(x) = e^{1/x} \cdot (-\frac{1}{x^2})$$

$$= -\frac{e^{1/x}}{x^2}$$

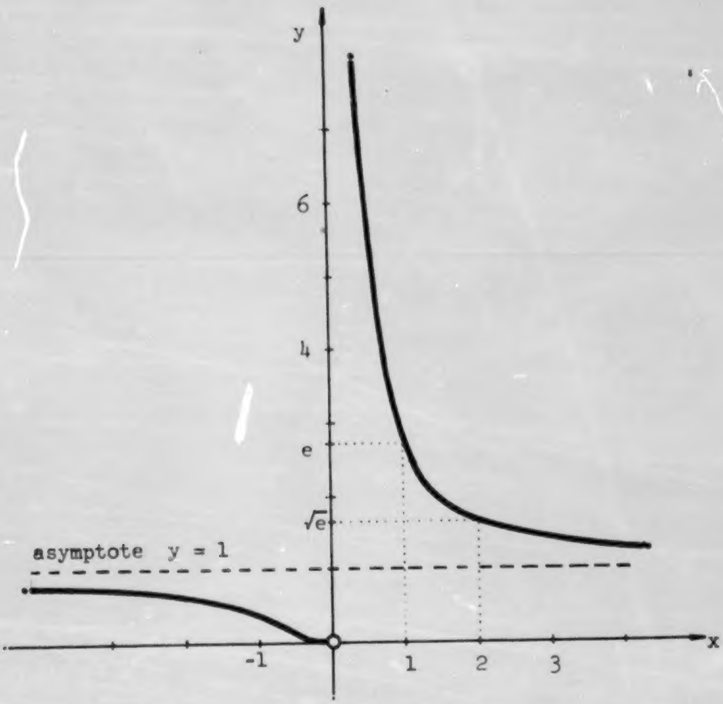
Thus $f'(x) < 0$ and f is a decreasing function whenever $x \neq 0$.

$$f''(x) = -e^{1/x} \cdot \frac{-2}{x^3} + \frac{-1}{x^2} \cdot (-\frac{e^{1/x}}{x^2})$$

$$= \frac{e^{1/x}}{x^4} (2x + 1)$$

$$f''(x) \geq 0 \text{ if } 2x + 1 \geq 0$$

Thus f is convex when $-\frac{1}{2} \leq x$ and f is concave when $x < -\frac{1}{2}$.



(c) $f : x \to \log_e \dfrac{1 + x^2}{1 - x^2} \qquad -1 < x < 1$

$$f'(x) = \frac{1 - x^2}{1 + x^2}\left[\frac{(1 - x^2)(2x) - (1 + x^2)(-2x)}{(1 - x^2)^2}\right]$$

$$= \frac{2x - 2x^3 + 2x + 2x^3}{(1 + x^2)(1 - x^2)}$$

$$= \frac{4x}{1 - x^4}$$

f is increasing when $0 \leq x < 1$.

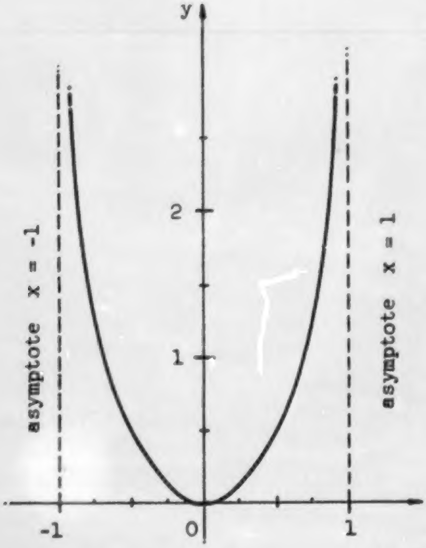
f is decreasing when $-1 < x < 0$.

$$f''(x) = \frac{(1 - x^4)(4) - (4x)(-4x^3)}{(1 - x^4)^2}$$

$$= \frac{4 - 4x^4 + 16x^4}{(1 - x^4)^2}$$

$$= \frac{4 + 12x^4}{(1 - x^4)^2} > 0$$

f is convex when $-1 < x < 1$.



8. (a) $y = xe^{-x^2}, \quad x = 0$

$$y' = x(e^{-x^2})(-2x) + e^{-x^2}$$

$$= e^{-x^2}(1 - 2x^2)$$

$$y'(0) = 1(1 - 0) = 1$$

$$y(0) = 0$$

The tangent line at $(0,0)$ is $y = x$.

(b) $y = e^{-11x^2}, \quad x = 1$

$$y(1) = e^{-11}$$

$$y' = -22x\, e^{-11x^2}$$

$$y'(1) = -22e^{-11}$$

The tangent line at $(1,e^{-11})$ is

$$y - e^{-11} = -22e^{-11}(x - 1)$$

$$y = -22e^{-11}x + 23e^{-11}$$

(c) $y = \sin(\pi - x^2)^{3/2}, \quad x = \sqrt{\pi}$

$$y(\pi) = 0$$

$$y' = \frac{3}{2}\sin(\pi - x^2)^{1/2} \cdot \cos(\pi - x^2) \cdot (-2x)$$

$$y'(\sqrt{\pi}) = 0$$

The tangent line at $(\sqrt{\pi},0)$ is $y = 0$.

(d) $y = \log_e(1 - x^2), \quad x = \frac{1}{2}$

$$y(\tfrac{1}{2}) = \log_e(\tfrac{3}{4})$$

$$y' = \frac{1}{1 - x^2} \cdot (-2x)$$

$$y'(\tfrac{1}{2}) = -\frac{4}{3}$$

The tangent line at $(\frac{1}{2}, \log_e \frac{3}{4})$ is

$$y - \log_e \frac{3}{4} = -\frac{4}{3}(x - \frac{1}{2}); \quad \log_e \frac{3}{4} \approx .2877$$

or

$$y \approx -\frac{4}{3}x + .9544$$

(e) $y = e^{e^x}, \quad x = 0$

$y(0) = e$

$y' = e^{e^x} \cdot e^x$

$y'(0) = e^1 \cdot 1 = e$

The tangent line at $(0,e)$ is

$$y - e = ex$$

or $$y = ex + e.$$

(f) $y = (e^x)^\pi, \ x = e$

$y(e) = e^{\pi e}$

$y' = \pi e^{\pi x}$

$y'(e) = \pi e^{\pi e}$

The tangent line at $(e, e^{\pi e})$ is

$$y - e^{\pi e} = \pi e^{\pi e}(x - e)$$

9. $f(x) = (Ax + B)\sin x + (Cx + D)\cos x$ and $f'(x) = x \sin x$.

$$f'(x) = A \sin x + (Ax + B)\cos x + C \cos x - (Cx + D)\sin x$$
$$= (A - Cx - D)\sin x + (C + Ax + B)\cos x$$

Two conditions must be met.

(i) $$(A - Cx - D)\sin x = x \sin x$$

or $$A - Cx - D = x$$

(ii) $$(C + Ax + B)\cos x = 0$$

or $$C + Ax + B = 0$$

We first observe in (i) that $C = -1$ and that $A = D$. In (ii) $A = 0$ is obvious and also $C = -B$. Thus $A = 0$, $B = +1$, $C = -1$, $D = 0$. Then $f(x) = \sin x - x \cos x$.

10. $g(x) = (Ax^2 + Bx + C)\sin x + (Dx^2 + Ex + F)\cos x$ and $g'(x) = x^2 \cos x$

$$g'(x) = (Ax^2 + Bx + C)\cos x + (2Ax + B)\sin x - (Dx^2 + Ex + F)\sin x + (2Dx + E)\cos x$$

$$= (2Ax + B - Dx^2 - Ex - F)\sin x + (2Dx + E + Ax^2 + Bx + C)\cos x$$

(i) $2Ax + B - Dx^2 - Ex - F = 0$

(ii) $2Dx + E + Ax^2 + Bx + C = x$

From (i) $D = 0$, $2A - E = 0$, and $B - F = 0$. Thus $D = 0$, $2A = E$, and $B = F$. Rewriting (ii) we have

$$2(0)x + 2A + Ax^2 + Bx + C = x^2.$$

It follows that $A = 1$, $B = 0$, and $C = -2A$. Thus $A = 1$, $B = 0$, $C = -2$, $D = 0$, $E = 2$, $F = 0$.

$$g(x) = (x^2 - 2)\sin x + 2x \cos x$$

Solutions Exercises 8-7

1. (a) $x \rightarrow \sqrt{\sin x}$

$$f'(x) = \frac{1}{2}(\sin x)^{-1/2} \cos x$$

or $$\frac{1}{2} \sqrt{\cot x \cos x}$$

(b) $x \rightarrow (\log_e x)^{\pi}$

$$f'(x) = \pi(\log_e x)^{\pi-1} \cdot \frac{1}{x}$$

$$= \frac{\pi}{x}(\log_e x)^{\pi-1}$$

(c) $s \rightarrow (s^3 + 3s)^{25}$

$$f'(s) = 25(s^3 + 3s)^{24} \cdot (3s^2 + 3)$$

(d) $t \rightarrow (e^t)^{-10}$

$$f'(t) = -10(e^t)^{-11} \cdot e^t$$

$$= -10e^{-10t}$$

(e) $x \rightarrow \dfrac{1}{\sqrt[3]{(1 - x)^2}} = (1 - x)^{-2/3}$

$$f'(x) = -\frac{2}{3}(1 - x)^{-5/3} \cdot (-1)$$

$$= \frac{2}{3}(1 - x)^{-5/3}$$

or $$= \frac{2}{3\sqrt[3]{(1 - x)^5}}$$

(f) $t \rightarrow (1 + \frac{1}{t})^{4/3}$

$$f'(t) = \frac{4}{3}(1 + \frac{1}{t})^{1/3} \cdot (-\frac{1}{t^2})$$

$$= -\frac{4}{3t^2}(1 + \frac{1}{t})^{1/3}$$

(g) $v \to \cos^{10} 2v$

$$f'(v) = 10 \cos^9 2v \cdot (-\sin 2v) \cdot (2)$$

$$= -20 \cos^9 2v \sin 2v$$

(h) $x \to \left(\int_0^x \sqrt{t^3 + 1}\, dt\right)^{1/2}$

$$f'(x) = \frac{1}{2}\left(\int_0^x \sqrt{t^3 + 1}\, dt\right)^{-1/2} \cdot \sqrt{x^3 + 1}$$

$$= \frac{\sqrt{x^3 + 1}}{2\left(\int_0^x \sqrt{t^3 + 1}\, dt\right)^{1/2}}$$

2. (a) $y = \dfrac{1}{1 - x^2}$

$$y' = \frac{-(-2x)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2}$$

(b) $y = \left(\dfrac{1}{1 - x^2}\right)^5 = \dfrac{1}{(1 - x^2)^5}$

$$y' = \frac{-5(1 - x^2)^4(-2x)}{(1 - x^2)^{10}}$$

$$= \frac{10x}{(1 - x^2)^6}$$

(c) $y = \dfrac{1}{1 + e^{2x}}$

$$y' = \frac{-2e^{2x}}{(1 + e^{2x})^2}$$

(d) $y = \dfrac{1}{1 + \log_e x}$

$$y' = \frac{-\frac{1}{x}}{(1 + \log_e x)^2}$$

(e) $y = \dfrac{1}{\sqrt{x + \frac{1}{x}}}$

$$y' = \frac{-\frac{1}{2}\,\frac{1}{\sqrt{x + \frac{1}{x}}} \cdot (1 - \frac{1}{x^2})}{x + \frac{1}{x}}$$

$$= \frac{1 - x^2}{2x^2(x + \frac{1}{x})^{3/2}}$$

(f) $y = (\sin x + \cos x)^{-1}$

$$y' = -\frac{(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$$

3. (a) $y = \sin^{3/2}(2x), \quad x = \frac{\pi}{6}$

$$y(\tfrac{\pi}{6}) = (\tfrac{\sqrt{3}}{2})^{3/2}$$

$$y'(x) = \tfrac{3}{2}\sin^{1/2} 2x \cdot \cos 2x \cdot 2$$

$$= 3\sin^{1/2} 2x \cos 2x$$

$$y'(\tfrac{\pi}{6}) = 3(\tfrac{\sqrt{3}}{2})^{1/2} \cdot \tfrac{1}{2}$$

$$= \tfrac{3}{2}(\tfrac{\sqrt{3}}{2})^{1/2}$$

The tangent line at $(\frac{\pi}{6}, (\frac{\sqrt{3}}{2})^{3/2})$ is

$$y - (\tfrac{\sqrt{3}}{2})^{3/2} = \tfrac{3}{2}(\tfrac{\sqrt{3}}{2})^{1/2}(x - \tfrac{\pi}{6})$$

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(b) $y = \left(\int_0^x e^{-t^2}\,dt\right)^2, \quad x = 0$

$y(0) = 0$

$y' = 2\left(\int_0^x e^{-t^2}\,dt\right)e^{-x^2}$

$y'(0) = 0$

The tangent line at $(0,0)$ is $y = 0$.

(c) $s = \sqrt{t + \frac{1}{t}}, \ t = 1$

$s(1) = \sqrt{2}$

$s' = \frac{1}{2}(t + \frac{1}{t})^{-1/2} \cdot (1 - \frac{1}{t^2})$

$s'(1) = 0$

The tangent line at $(1,\sqrt{2})$ is $s = \sqrt{2}$.

4. (a) $y = \dfrac{1}{1 + x^2}$

(i) y is defined for all x.

(ii) $y' = \dfrac{-2x}{(1 + x^2)^2} = -2x(1 + x^2)^{-2}$

y is increasing when $x \leq 0$.

y is decreasing when $0 < x$.

(iii) $y'' = (-2x)(-2)(1 + x^2)^{-3}2x + -2(1 + x^2)^{-2}$

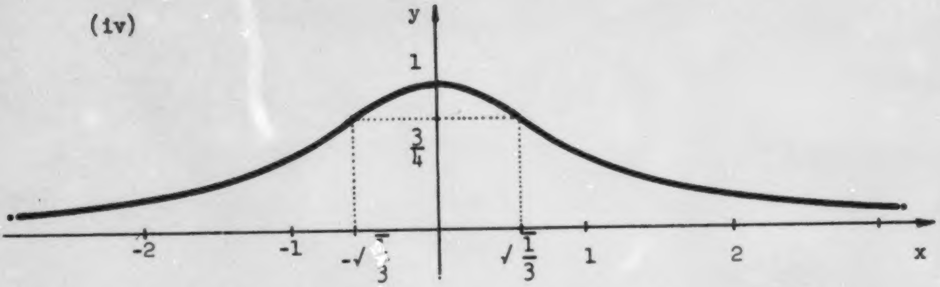
$= \dfrac{6x^2 - 2}{(1 + x^2)^3}$

y is convex when $\sqrt{\frac{1}{3}} \leq |x|$

y is concave when $|x| < \sqrt{\frac{1}{3}}$

(iv) Horizontal asymptote of $y = 0$.

(iv)



(b) $y = \sqrt{\sin x}$

(i) y is defined when $2n\pi \le x \le (2n + 1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(ii) $y' = \dfrac{1 \cos x}{2\sqrt{\sin x}} = \dfrac{1}{2} \cos x \sin^{-1/2} x$

y is increasing when $2n\pi \le x < (2n + \frac{1}{2})\pi$.

y is decreasing when $(2n + \frac{1}{2})\pi \le x \le (2n + 1)\pi$.

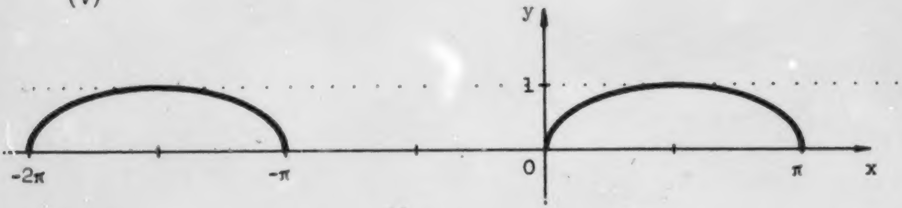
(iii) $y'' = \frac{1}{2}(\cos x)(-\frac{1}{2})(\sin^{-3/2} x \cos x)$

$$= \frac{-(\cos^2 x + 2 \sin^2 x)}{4 \sin^{3/2} x}$$

y is everywhere concave.

(iv) There are no asymptotes.

(v)



5. (a) $x \rightarrow \dfrac{1}{1 - e^x}$, $x > 0$

$$f'(x) = \frac{-(-e^x)}{(1 - e^x)^2}$$

$$= \frac{e^x}{(1 - e^x)^2}$$

$f'(x) > 0$ thus f is an increasing function.

(h) $D(\csc 3x)^{1/6} = \frac{1}{6}(\csc 3x)^{-5/6} \cdot (-\csc 3x \cot 3x)(3)$

$$= -\frac{1}{2}(\csc 3x)^{1/6} \cot 3x$$

(i) $D[\sec(\csc x)] = [\sec(\csc x)\tan(\csc x)][-\csc x \cot x]$

7. $f : x \to \sec x$, f is not defined for $x = (n + \frac{1}{2})\pi$, $n = 0, \pm 1, \pm 2, \ldots$

$f' : x \to \sec x \tan x = \dfrac{\sin x}{\cos^2 x}$

$f'(x) \geq 0$ when $\sin x > 0$ or when $2n\pi \leq x < (2n + 1)\pi$, $x \neq (n \pm \frac{1}{2})\pi$

$f''(x) = \sec x \sec^2 x + \tan x \sec x \tan x$

$$= \sec x (\sec^2 x + \tan^2 x) = \frac{\sec^2 x + \tan^2 x}{\cos x}$$

$f''(x) \geq 0$ when $\cos x \geq 0$.

Thus f is convex when $(2n - \frac{1}{2})\pi < x < (2n + \frac{1}{2})\pi$.

8. (a) $D(\sec x \csc x) = -\sec x \cot x \csc x + \csc x \tan x \sec x$

$$= -\sec x \frac{\csc x}{\sec x} \csc x + \csc x \frac{\sec x}{\csc x} \sec x$$

(i) $= -\csc^2 x + \sec^2 x$

(ii) $= -1 - \cot^2 x + 1 + \tan^2 x$

$= -\cot^2 x + \tan^2 x$

(iii) $= -4 \csc 2x \cot 2x$

(b) (i) $D(\tan x \cot x) = D(1) = 0$

(ii) $D(\sin x \csc x) = D(1) = 0$

(iii) $D(\cos x \sec x) = D(1) = 0$

(c) (i) $D(\sin x \cot x) = D(\cos x)$

$= -\sin x$

(ii) $D(\cos x \tan x) = D(\sin x)$

$= \cos x$

(b) $x \to (x^3 + 3x)^{10}, \quad x \geq 0$

$$f'(x) = 10(x^3 + 3x)^9(3x^2 + 3)$$
$$= 30(x^2 + 1)(x^3 + 3x)^9$$

$f'(x) \geq 0$ thus f is an increasing function.

6. (a) $y = \sec x = \dfrac{1}{\cos x}$

$$y' = \frac{-(-\sin x)}{\cos^2 x}$$
$$= \tan x \sec x$$

(b) $y = \csc x = \dfrac{1}{\sin x}$

$$y' = \frac{-\cos x}{\sin^2 x}$$
$$= -\cot x \csc x$$

(c) $y = \tan x = \dfrac{\sin x}{\cos x} = \sin x(\cos x)^{-1}$

$$y' = \sin x(-1)(\cos x)^{-2}(-\sin x) + \cos x(\cos x)^{-1}$$
$$= \frac{\sin^2 x}{\cos^2 x} + 1$$
$$= \tan^2 x + 1 = \sec^2 x$$

(d) $y = \cot x = \dfrac{\cos x}{\sin x} = \cos x(\sin x)^{-1}$

$$y' = \cos x(-1)(\sin x)^{-2}\cos x + (-\sin x)(\sin x)^{-1}$$
$$= -\frac{\cos^2 x}{\sin^2 x} - 1$$
$$= -(\cot^2 x + 1)$$
$$= -\csc^2 x$$

(e) $D(\tan 3x) = 3 \sec^2 3x$

(f) $D\sqrt{\tan 2x} = \dfrac{2 \sec^2 2x}{2\sqrt{\tan 2x}}$

(g) $D(\sec^2 x^2) = 2 \sec x^2 \cdot (\sec x^2 \tan x^2)(2x)$

$$= 4x \sec^2 x^2 \tan x^2$$

9. (a) $$D\left(\frac{\tan^{(k+1)}x}{k+1}\right) = (k+1)\cdot\frac{\tan^{((k+1)-1)}x}{k+1}\cdot D(\tan x)$$

$$= \tan^{k} x \sec^{2} x$$

(b) $$D\left(\frac{1}{k}\csc^{k} x\right) = k\left[\frac{1}{k}\csc^{(k-1)} x\right]D(\csc x)$$

$$= \csc^{(k-1)} x \csc x \cot x$$

$$= \csc^{k} x \cot x$$

(c) $$D(\cot^{2} x) = D(\csc^{2} x - 1)$$

$$= D(\csc^{2} x) + D(-1)$$

$$= D(\csc^{2} x)$$

10. (a) $$\frac{u}{v} = u\cdot\frac{1}{v}$$

$$\left(\frac{u}{v}\right)' = u\cdot\left(\frac{1}{v}\right)' + u'\left(\frac{1}{v}\right)$$

$$= u\cdot\frac{-v'}{v^{2}} + u'\cdot\frac{1}{v}$$

$$= \frac{vu' - uv'}{v^{2}}$$

(b) $$D\left(\frac{x^{2}+1}{3x^{2}-x}\right) = \frac{(3x^{2}-x)(2x) - (x^{2}+1)(6x-1)}{(3x^{2}-x)^{2}}$$

$$= \frac{-x^{2} - 6x + 1}{(3x^{2}-x)^{2}}$$

Solutions Exercises 8-8

1. (a) $$D\left(\frac{x}{x-1}\right) = \frac{(x-1)(1) - (x)(1)}{(x-1)^2}$$

$$= \frac{-1}{(x-1)^2}$$

(b) $$D\left(\frac{x^2}{1+x^2}\right) = \frac{(1+x^2)(2x) - (x^2)(2x)}{(1+x^2)^2}$$

$$= \frac{1}{(1+x^2)^2}$$

(c) $$D\left(1 - \frac{1}{x}\right)^{-1} = D\left(\frac{x}{x-1}\right)$$

$$= \frac{-1}{(x-1)^2} \quad \text{from part (a).}$$

(d) $$D\left(\frac{3+2x^2}{2-x^2}\right) = \frac{(2-x^2)(4x) - (3+2x^2)(-2x)}{(2-x^2)^2}$$

$$= \frac{14x}{(2-x^2)^2}$$

(e) $$D\left(\frac{1}{x} + \frac{1}{1-x}\right) = D\left(\frac{1-x+x}{x-x^2}\right)$$

$$= D\left(\frac{1}{x-x^2}\right)$$

$$= \frac{(1-2x)}{(x-x^2)^2}$$

$$= \frac{2x-1}{(x-x^2)^2}$$

(f) $$D\left(\frac{\sqrt{x}}{1+x^2}\right) = \frac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-4x^2}{2\sqrt{x}\,(1+x^2)^2}$$

$$= \frac{1-3x^2}{2\sqrt{x}\,(1+x^2)^2}$$

(g) $$D\left(\frac{1}{1+\sqrt{x}}\right) = \frac{-\frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2}$$

$$= \frac{-1}{2\sqrt{x}\,(1+\sqrt{x})^2}$$

(h) $$D\left(\frac{x^2-1}{x^2+1}\right)^{-1} = D\left(\frac{x^2+1}{x^2-1}\right)$$

$$= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

(i) $$D\left(\frac{\sin x}{1+\tan x}\right) = \frac{(1+\tan x)\cos x - \sin x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\cos x + \sin x - \sin x \sec^2 x}{(1+\tan x)^2}$$

(j) $$D\left(\frac{e^x}{1+x^2}\right) = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{e^x(x^2-2x+1)}{(1+x^2)^2}$$

$$= \frac{e^x(x-1)^2}{(1+x^2)^2}$$

(k) $$D\left(\frac{x\log_e x}{1-2x}\right) = \frac{(1-2x)(x\cdot\frac{1}{x} + \log_e x) - (x\log_e x)(-2)}{(1-2x)^2}$$

$$= \frac{1-2x+\log_e x}{(1-2x)^2}$$

(ℓ) $D(\cos x \sec x) = D(1) = 0$

(m) $$D\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2}$$

(n) $$D[(1 + \frac{1}{x})(1 + \log_e x)] = (1 + \frac{1}{x})\frac{1}{x} + (1 + \log_e x)\left(- \frac{1}{x^2}\right)$$

$$= (1 + \frac{1}{x})\frac{1}{x} + (1 + \log_e x)\left(- \frac{1}{x^2}\right)$$

$$= \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} \log_e x$$

$$= \frac{1}{x^2}(x - \log_e x)$$

(o) $$D\left(\frac{\log_e x^2}{\sqrt{x^2 + 1}}\right) = D\left(\frac{2 \log_e x}{\sqrt{x^2 + 1}}\right)$$

$$= \frac{\sqrt{x^2 + 1} \cdot \frac{2}{x} - (2 \log_e x) \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x}{x^2 + 1}$$

$$= \frac{\frac{2}{x\sqrt{x^2 + 1}} [(x^2 + 1) - x^2 \log_e x]}{(x^2 + 1)}$$

$$= \frac{2(x^2 + 1 - x^2 \log_e x}{x(x^2 + 1)^{3/2}}$$

2. $$D(\cot x) = D(\frac{\cos x}{\sin x})$$

$$= \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

3. (a) $$y = \frac{x + 2}{x^2 - 1}$$

This function is not defined for $x = \pm 1$. To the left of $x = -1$, $y \to +\infty$ and to the right of $x = -1$, $y \to -\infty$. To the left of $x = +1$, $y \to -\infty$ and to the right of $x = +1$, $y \to +\infty$. This describes the vertical asymptotes at $x = +1$ and $x = -1$. At $x = -2$, $y = 0$ and at $x = 0$, $y = -2$.

$$y = \frac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \frac{\frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$$

As $x \to +\infty$, $y \to 0^+$ and as $x \to -\infty$, $y \to 0^-$. This describes the horizontal asymptote, $y = 0$.

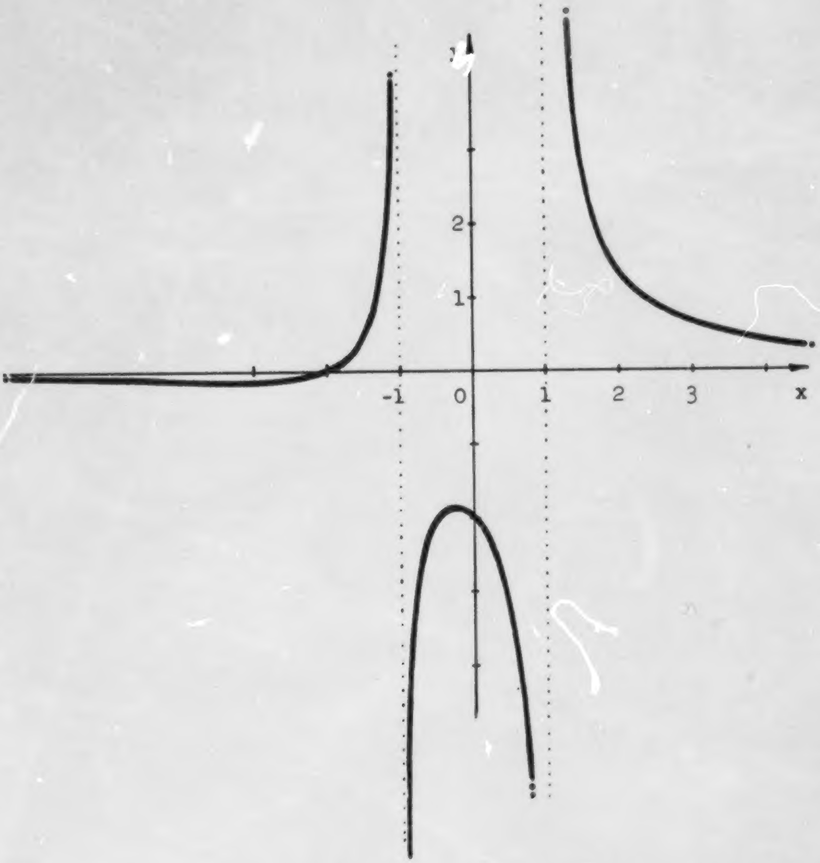
$$y' = \frac{(x^2 - 1)(1) - (x + 2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{-x^2 - 4x - 1}{(x^2 - 1)^2} = \frac{-(x^2 + 4x + 1)}{(x^2 - 1)^2}$$

$y' = 0$ if $x = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$.

There are two extremum points, one is a minimum point when $x = -2 - \sqrt{3} \approx -3.732$ and the other is a maximum point when $x = -2 + \sqrt{3} \approx -.268$. We see that y decreases for $x < -2 - \sqrt{3}$, increases for $-2 - \sqrt{3} \leq x < -2 + \sqrt{3}$, and finally decreases again for $-2 + \sqrt{3} \leq x$.

The discussion of y'' is rather tedius and does not greatly increase our understanding of the function.



(b) $y = \frac{x - 1}{x + 1}$

y is undefined for $x = -1$. As $x \to -1$ from the left $y \to +\infty$ and as $x \to -1$ from the right $y \to -\infty$. This describes the vertical asymptote at $x = -1$.

There is a zero at $x = 1$.

Rewriting $y = \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}}$ we see that as $x \to +\infty$, $y \to 1$ and as $x \to -\infty$, $y \to 1$. Thus, there is a horizontal asymptote of $y = 1$.

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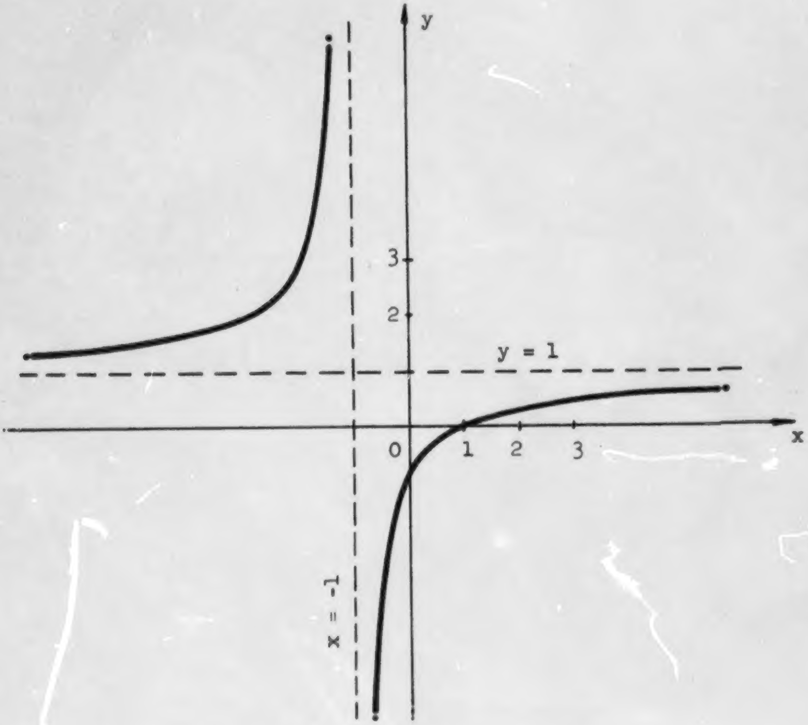
$$y' = \frac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2}$$

$$= \frac{2}{(x + 1)^2}$$

y' is always positive and y is increasing.



(c) $y = \frac{e^{-2x}}{1 + x}$, y is undefined at $x = -1$.

As $x \to -1$ from the left $y \to -\infty$ and as $x \to -1$ from the right $y \to +\infty$. This describes the vertical asymptote at $x = -1$.

When $x \to +\infty$, $y \to \frac{0}{\infty} = 0$.

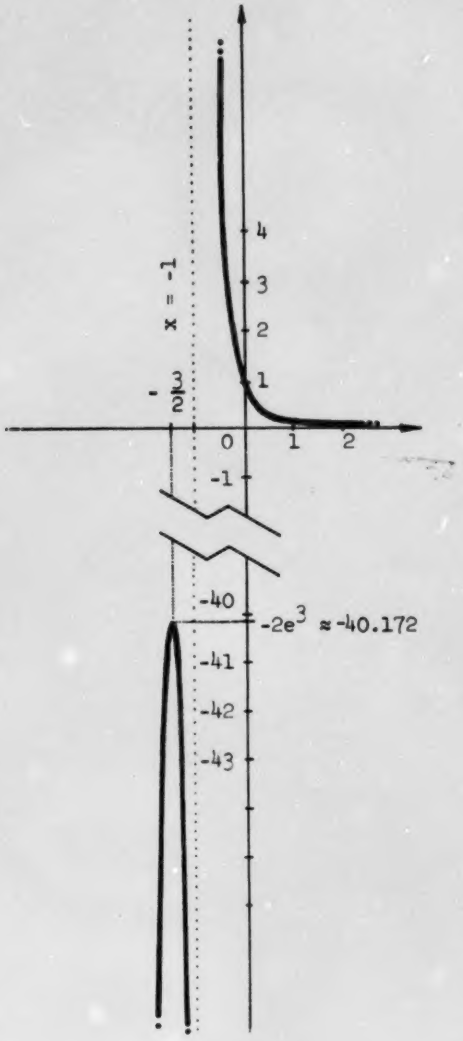
From previous discussions in Chapter 5 we know that $\frac{e^{|x|}}{|x|}$ is greater than any predetermined number when x is large enough. It follows that as $x \to -\infty$ we have $e^{-2x} > 0$ and $1 + x < 0$, thus $y \to -\infty$. There is an asymptote for $x \to +\infty$ which is $y = 0$, but none for $x \to -\infty$.

$$y' = \frac{(1 + x)(-2e^{-2x}) - e^{-2x}}{(1 + x)^2}$$

$$= \frac{-e^{-2x}(3 + 2x)}{(1 + x)^2}$$

$$y' \geq 0 \text{ if } 3 + 2x \leq 0 \text{ or } x \leq -\frac{3}{2}$$

Thus y is increasing when $-x \leq -\frac{3}{2}$ and decreasing when $-\frac{3}{2} < x$. It appears that $x = -\frac{3}{2}$ is a maximum.



4. (a) $\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_0^{\pi/4}$

$= 1 - 0 = 1$

(b) $\int_{-\pi/3}^{0} \sec x \tan x \, dx = \sec x \Big|_{-\pi/3}^{0}$

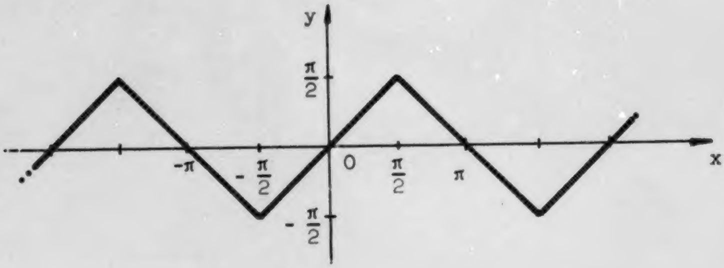
$= 1 - (+2) = -1$

Solutions Exercises 8-9

1. (a) $f : x \rightarrow \arcsin(\sin x)$

Domain: the set of all real numbers.

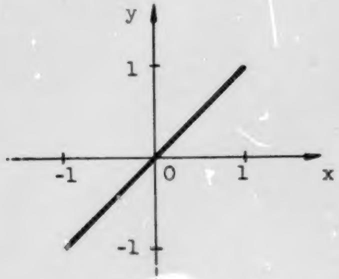
Range: the set of all real numbers $y = f(x)$ such that $-\frac{\pi}{2} \leq y \leq 2$.



(b) $f(x) = \sin(\arcsin x)$

Domain: all x such that $-1 \leq x \leq 1$.

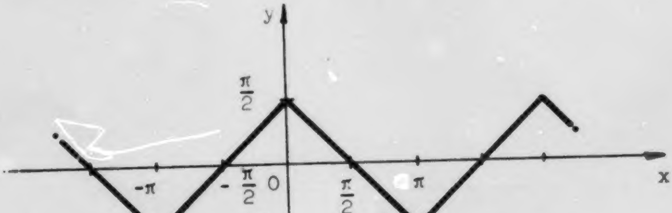
Range: all y such that $-1 \leq y \leq 1$.



(c) $f(x) = \arcsin(\cos x)$

Domain: set of all real numbers.

Range: all $y = f(x)$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

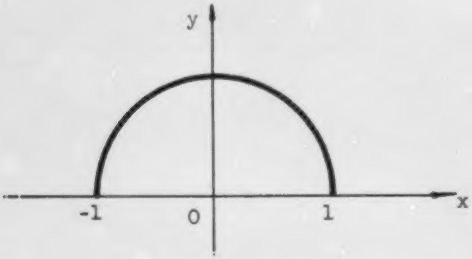


(d) $f(x) = \cos(\arcsin x)$

Domain: all x where $-1 \leq x \leq 1$.

Range: all $y = f(x)$ where $0 \leq y \leq 1$.

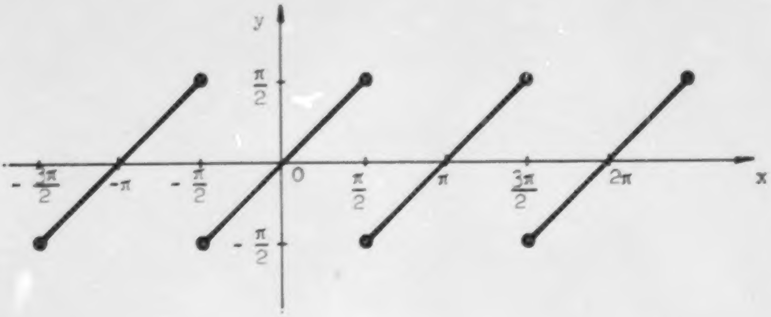
The graph is the semicircle with center at origin, radius = 1, and $y \geq 0$.



(e) $f(x) = \arctan(\tan x)$

Domain: all real x except $x = (2n + 1)\frac{\pi}{2}$, n an integer.

Range: all $y = f(x)$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.



2. Let $g : x \to \arccos x$ and $f : y \to \cos y$, where $y = \arccos x$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$.

Then $g'(x) = \frac{1}{f'(y)}$ (for $f'(y) \neq 0$, i.e., for $-\sin y \neq 0$) for $0 < y < \pi$ and hence for $-1 < x < 1$.

$$\therefore \quad D \arccos x = g'(x) = \frac{1}{-\sin y}, \quad \text{where } \sin y > 0,$$

$$= \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

3. (a) Let $g : x \rightarrow \text{arccot } x$ and $f : y \rightarrow \cot y$ where $y = \text{arccot } x$ for all real x and $0 < y < \pi$.

$$g'(x) = \frac{1}{f'(y)} = \frac{1}{-\csc^2 y} = \frac{-1}{1 + \cot^2 y}, \quad 0 < y < \pi,$$

$$= \frac{-1}{1 + x^2}, \quad \text{for all } x.$$

(b) Let $g : x \rightarrow \text{arcsec } x$ and $f : y \rightarrow \sec y$ where $y = \text{arcsec } x$, $|x| \geq 1$ and $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$.

Then $g'(x) = \frac{1}{f'(y)}$

$$= \frac{1}{\sec y \tan y}, \quad 0 < y < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < y < \pi$$

$$= \frac{1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1$$

Note: $\frac{1}{\sec y \tan y} = \frac{\cos^2 y}{\sin y} > 0$ for $0 < y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y < \pi$, hence $g'(x) > 0$ for all x such that $|x| > 1$.

(c) Let $g(x) = \text{arccsc } x$ and $f(y) = \csc y$ where $y = \text{arccsc } x$ for $|x| \geq 1$ and $0 < |y| \leq \frac{\pi}{2}$.

Then $g'(x) = \frac{1}{f'(y)}$

$$= \frac{1}{-\csc y \cot y}, \quad 0 < |y| < \frac{\pi}{2}$$

$$= \frac{-1}{|x|\sqrt{x^2 - 1}}, \quad |x| > 1.$$

Note: $g'(x) < 0$ since $\frac{1}{\csc y \cot y} = \frac{-\sin^2 y}{\cos y} < 0$ for $0 < |y| < \frac{\pi}{2}$.

4. (a) $D(\arcsin x + \arccos x) = \dfrac{1}{\sqrt{1 - x^2}} + \dfrac{-1}{\sqrt{1 - x^2}} = 0$

Note that $\arcsin x + \arccos x = \frac{\pi}{2}$ and $D(\frac{\pi}{2}) = 0$.

(b)
$$D(x^2 \arcsin x) = x^2 \cdot \frac{1}{\sqrt{1 - x^2}} + 2x \arcsin x$$
$$= x\left(\frac{x}{\sqrt{1 - x^2}} + 2 \arcsin x\right)$$

(c)
$$D\left(\frac{x^2}{\arctan x}\right) = \frac{2x \arctan x - \dfrac{x^2}{1 + x^2}}{(\arctan x)^2}$$
$$= \frac{2x(1 + x^2) \arctan x - x^2}{(1 + x^2)(\arctan x)^2}$$

(d)
$$D(\arcsin x)^3 = \frac{3(\arcsin x)^2}{\sqrt{1 - x^2}}$$

(e)
$$D\left(\frac{1}{1 + \arcsin x}\right) = \frac{-1}{\sqrt{1 - x^2}(1 + \arcsin x)^2}$$

(f)
$$D\left(\frac{1 - \arctan x}{1 + \arctan x}\right) = \frac{(1 + \arctan x)\dfrac{-1}{1 + x^2} - (1 - \arctan x)\dfrac{1}{1 + x^2}}{(1 + \arctan x)^2}$$
$$= \frac{-2}{(1 + x^2)(1 + \arctan x)^2}$$

5. Let $f : x \to \arcsin x$.

Then $\dfrac{f(x + h) - f(x)}{h}$, (for $x = 0$)
$$= \frac{\arcsin h - \arcsin 0}{h}$$
$$= \frac{\arcsin h}{h}.$$
$$\therefore \lim_{h \to 0} \frac{\arcsin h}{h} = f'(0) = \frac{1}{\sqrt{1 - 0^2}} = 1.$$

6. (a) $y = \arcsin x^2$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$$

(b) $y = \arctan (3x + 2)$

$$\frac{dy}{dx} = \frac{3}{9x^2 + 12x + 5}$$

(c) $y = e^{\arcsin x}$

$$\frac{dy}{dx} = \frac{e^{\arcsin x}}{\sqrt{1 - x^2}}$$

(d) $y = e^{2x} \arcsin \frac{1}{x}$

$$\frac{dy}{dx} = e^{2x} \cdot \frac{-1}{\sqrt{1 - \frac{1}{x^2}}} \cdot \frac{-1}{x^2} + 2e^{2x} \arcsin \frac{1}{x}$$

$$= e^{2x} \left(\frac{-1}{x^2 \sqrt{x^2 - 1}} + 2 \arcsin \frac{1}{x}\right)$$

7. (a) $$\int_0^1 \frac{1}{1 + x^2}\, dx = \arctan x \Big|_0^1$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(b) $$\int_{-\pi/4}^{\pi/6} \frac{1}{\sqrt{1 - t^2}}\, dt = \arcsin t \begin{array}{l} \pi/6 \approx 0.5236 \\ -\pi/4 \approx -0.7854 \end{array}$$

$$\approx \arcsin(0.5236) - \arcsin(-0.7854)$$

$$\approx 0.551 - (-0.895) \approx 1.45$$

8. (a) $$F(x) = \int_0^x \frac{2}{1 + t^2}\, dt$$

$$F'(x) = \frac{2}{1 + x^2}$$

(b) $$F(x) = \int_0^{x^3} \frac{3}{\sqrt{1 - t^2}}\, dt$$

$$F'(x) = \frac{9x^2}{\sqrt{1 - x^6}}$$

(c) $$F(x) = \int_0^{\sin x} \frac{1}{1 + t^2}\, dt$$

$$F'(x) = \frac{\cos x}{1 + \sin^2 x}$$

9. $$\lim_{n \to \infty} \int_0^n \frac{1}{1 + t^2}\, dt = \lim_{n \to \infty} \arctan \Big|_0^n$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

10. (a) $$g : x \to \frac{1 - x}{1 + x}, \quad x > -1$$

$$f : x \to \frac{1 - x}{1 + x}$$

$$f'(x) = \frac{(1 + x)(-1) - (1 - x)(1)}{(1 + x)^2}$$

$$f'(x) = \frac{-2}{(1 + x)^2}$$

(b) $$g : x \to x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

$$f : x \to \frac{|x|}{x} \sqrt{|x|} = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

$$f' : x = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x \geq 0 \\ -\frac{-1}{2\sqrt{-x}} & \text{if } x < 0 \end{cases}$$

$$f' : x = \frac{1}{2\sqrt{|x|}}$$

11. If f and g are inverses let $g(x) = c$. But then $f(c) = x$ or by substitution $f(g(x)) = x$.

The derivative $D(f(g(x))) = f'(g(x))g'(x) = 1$.

Then
$$g'(x) = \frac{1}{f'(g(x))} .$$

The only difference is that rule (5) was developed for a strictly increasing function, that is $f'(g(x)) > 0$.

12. f_1 and f_2 are the inverses of g_1 and g_2 respectively and $g(x) = g_1(g_2(x))$.

(a) Since
$$f_1(g_1(x)) = x$$
then
$$\begin{aligned} f_1(g(x)) &= f_1(g_1(g_2(x))) \\ &= g_2(x). \end{aligned}$$
Since
$$f_2(g_2(x)) = x$$
then
$$\begin{aligned} f_2(f_1(g(x))) &= f_2(g_2(x)) \\ &= x. \end{aligned}$$

It follows that $f_2(f_1(x))$ is the inverse of $g(x)$.

(b) $x \to (3x + 2)^2, \quad x \geq -\frac{2}{3}$

Let $g_1 : x \to x^2$ and $g_2 : x \to 3x + 2$.

We find that $f_1 : x \to \sqrt{x}$ and $f_2 : x \to \frac{x - 2}{3}$.

Finally,
$$\begin{aligned} f(x) &= f_2(f_1(x)) = f_2(\sqrt{x}) \\ &= \frac{\sqrt{x} - 2}{3} . \end{aligned}$$

(c)
$$\begin{aligned} f'(x) &= D\left(\frac{\sqrt{x}}{3} - \frac{2}{3}\right) \\ &= \frac{1}{6\sqrt{x}} \end{aligned}$$

13. f and g are inverses.

$$y = g(x) \quad \text{and} \quad x = f(y)$$

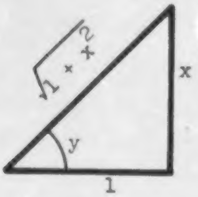
$$\left.\frac{dx}{dy}\right|_{y=a} = f'(a)$$

$$\left.\frac{dy}{dx}\right|_{x=f(a)} = g'(f(a)) = \frac{1}{f'(a)}\ , \qquad \text{by (5).}$$

Thus
$$\left.\frac{dx}{dy}\right|_{y=a} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=f(a)}}\ .$$

14. (a) $y = \arctan x$ means that $x = \tan y$.

Thus $\frac{dx}{dy} = \sec^2 y$ and $\frac{dy}{dx} = \frac{1}{\sec^2(\arctan x)}$



$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(b) If $y = \log_e x$ then $x = e^y$. Thus $\frac{dx}{dy} = e^y = e^{\log_e x} = x$ and $\frac{dy}{dx} = \frac{1}{x}$.

(c) If $y = \sqrt{x}$ then $x = y^2$. Then $\frac{dx}{dy} = 2y = 2\sqrt{x}$ and $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

(d) If $y = x^\pi$ then $x = \sqrt[\pi]{y}$ provided that $x \geq 0$.

Thus $\frac{dx}{dy} = \frac{1}{\pi}\, y^{(1/\pi\ -1)} = \frac{1}{\pi}(x^\pi)^{(1/\pi\ -1)} = \frac{1}{\pi}\, x^{1-\pi}$

and $\frac{dy}{dx} = \frac{\pi}{x^{1-\pi}} = \pi x^{\pi-1}$.

Solutions Exercises 8-10

1. $y = x^r$, where $r = \frac{p}{q}$, $x > 0$. If $y^q = x^p$, $Dy^q = Dx^p$ and

$$qy^{q-1}\,Dy = px^{p-1}, \quad \text{whence} \quad Dy = \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,x^{p/q-1} = rx^{r-1}.$$

2. (a) $5x^2 + y^2 = 12$

$10x + 2y\,y' = 0$ and $y' = \frac{-5x}{y}$

(b) $2x^2 - y^2 + x - 4 = 0$

$4x - 2y\,y' + 1 = 0$

$y = \frac{4x + 1}{2y}$

(c) $y^2 - 3x^2 + 6y = 12$

$2y\,y' - 6x + 6\,y' = 0$

$y' = \frac{3x}{y + 3}$

(d) $x^3 + y^3 - 2xy = 0$

$3x^2 + 3y^2\,y' - 2x\,y' - 2y = 0$

$y' = \frac{2y - 3x^2}{3y^2 - 2x}$

3. (a) $x^2 = \frac{y - x}{y + x}$

$$2x = \frac{(y + x)(Dy - 1) - (y - x)(Dy + 1)}{(y + x)^2}$$

$$Dy = \frac{x(y + x)^2 + y}{x}$$

(b) $x^2y + xy^2 = x^3$

$2xy + x^2\,Dy + y^2 + 2xy\,Dy = 3x^2$

$$Dy = \frac{3x^2 - 2xy - y^2}{x^2 + 2xy}$$

(c) $x^m y^n = 10$, (m, n integers)

$$mx^{m-1} y^n + nx^m y^{n-1} Dy = 0$$

$$Dy = \frac{-my}{nx}$$

(d) $\sqrt{xy} + x = y^{-1}$

$$\frac{1}{2}(xy)^{-1/2}(x\ Dy + y) + 1 = -\frac{Dy}{y^2}$$

$$Dy = \frac{-2y^2\sqrt{xy} - y^3}{xy^2 + 2\sqrt{xy}}$$

4. (a) $x\sqrt{y} + y\sqrt{x} = a\sqrt{a}$, a constant

$$\sqrt{y}\frac{dx}{dy} + \frac{x}{2\sqrt{y}} + \frac{y}{2\sqrt{x}}\frac{dx}{dy} + \sqrt{x} = 0$$

$$\frac{dx}{dy} = \frac{-x(\sqrt{x} + 2\sqrt{y})}{y(2\sqrt{x} + \sqrt{y})}$$

(b) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$

$$4x\frac{dx}{dy} + 3y\frac{dx}{dy} + 3x + 2y + \frac{dx}{dy} - 2 = 0$$

$$\frac{dx}{dy} = \frac{2 - 3x - 2y}{4x + 3y + 1}$$

(c) $(x + y)^{1/2} + (x - y)^{1/2} = 4$

$$\frac{1}{2(x + y)^{1/2}}\left(\frac{dx}{dy} + 1\right) + \frac{1}{2(x - y)^{1/2}}\left(\frac{dx}{dy} - 1\right) = 0$$

$$\frac{dx}{dy} = \frac{\sqrt{x + y} - \sqrt{x - y}}{\sqrt{x + y} + \sqrt{x - y}}$$

(d) $3x^2 + x^2y^2 = y^4 + 5$

$$6x\frac{dx}{dy} + 2xy^2\frac{dx}{dy} + 2x^2y = 4y^3$$

$$\frac{dx}{dy} = \frac{2y^3 - x^2y}{3x + xy^2}$$

5. (a) $2x^2 + 3xy + y^2 + x - 2y + 1 = 0$ at the point $(-2,1)$.

$4x + 3y + 3x\,y' + 2y\,y' + 1 - 2y' = 0$

At $(-2,1)$, $y' = -\frac{2}{3}$.

(b) $x^3 + y^2x^2 + y^3 - 1 = 0$ at the point $(1,-1)$.

$3x^2 + 2yx^2\,y' + 2xy^2 + 3y^2\,y' = 0$

At $(1,-1)$, $y' = -5$.

(c) $x^2 - x\sqrt{xy} - 6y^2 = 2$ at the point $(4,1)$.

$2x - \frac{3}{2}\sqrt{xy} - \frac{1}{2}x^{3/2}\,y^{-1/2}\,y' - 12y\,y' = 0$

At $(4,1)$, $y' = \frac{5}{16}$.

(d) $x \cos y = 3x^2 - 5$ at the point $(\sqrt{2}, \frac{\pi}{4})$

$x(-\sin y\, y') + \cos y = 6x$

At $(\sqrt{2}, \frac{\pi}{4})$, $y' = -\frac{11\sqrt{2}}{2}$

6. (a) $x^3 - 3axy + y^3 = 0$

$3x^2 - 3ax\,Dy - 3ay + 3y^2\,Dy = 0$

$$Dy\Big|_{x=y} = \left.\frac{-3x^2 + 3ay}{-3ax + 3y^2}\right|_{x=y\neq0} = -1$$

(b) $x^m + y^m = 2$

$mx^{m-1} + my^{m-1}\,Dy = 0$

$$Dy\Big|_{x=y} = \left.\frac{-mx^{m-1}}{my^{m-1}}\right|_{x=y\neq0} = -1$$

(c) $x^2 + y^2 = 2axy + a^2$

$2x + 2y\,Dy = 2ax\,Dy + 2ay$

$$Dy\Big|_{x=y} = \left.\frac{2ay - 2x}{2y - 2ax}\right|_{x=y} = -1$$

All three curves are symmetric about the line $y = x$. Thus at the point where $x = y$ the tangents to the curves are orthogonal to the line.

7. (a) $a \sin y + b \cos x = 0$ (a, b constant).

$$a \cos y \, y' - b \sin x = 0$$

$$y' = \frac{b \sin x}{a \cos y}$$

(b) $x \cos y + y \sin x = 0$

$$\cos y - x \sin y \, y' + \sin x \, y' + y \cos x = 0$$

$$y' = \frac{\cos y + y \cos x}{x \sin y - \sin x}$$

(c) $\sin xy = \sin x + \sin y$

$$(\cos xy)(y + x y') = \cos x + \cos y \, y'$$

$$y' = \frac{y \cos xy - \cos x}{\cos y - x \cos xy}$$

(d) $\csc(x + y) = y$

$$-\csc(x + y) \cot(x + y)(1 + y') = y'$$

$$y' = -\frac{\csc(x + y) \cot(x + y)}{\csc(x + y) \cot(x + y) + 1}$$

(e) $x \tan y - y \tan x = 1$

$$\tan y + x \sec^2 y \, y' - \tan x \, y' - y \sec^2 x = 0$$

$$y' = \frac{\tan y - y \sec^2 x}{\tan x - x \sec^2 y}$$

(f) $\tan xy - x^2 = 0$

$$(\sec^2 xy)(y + x y') - 2x = 0$$

$$y' = \frac{2x - y \sec^2 xy}{x \sec^2 xy}$$

(g) $y \sin x = x \tan y$

$$\sin x \, y' + y \cos x = \tan y + x \sec^2 y \, y'$$

$$y' = \frac{\tan y - y \cos x}{\sin x - x \sec^2 y}$$

8. $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\, y' = 0$

$y' = -\frac{\sqrt{y}}{\sqrt{x}}$ which is always negative since $x, y > 0$.

Teacher's Commentary

Appendix 3

MATHEMATICAL INDUCTION

TC A3-1. The Principle of Mathematical Induction

The Principle of Mathematical Induction may be thought of as a postulate for the set of natural numbers N, rather than as a postulate about legitimate methods of proof (Metamathematics). Thus, we may state the principle in the following form:

Let M be a subset of N satisfying

(i) $1 \in M$,

(ii) if $n \in M$, then $n + 1 \in M$,

then $M = N$.

We can then deduce the form of the principle in the text by setting M equal to the set of natural numbers n for which A_n is true.

As stated here, the Principle of Mathematical Induction can be used to play a central role in the axiomatic development of the natural numbers. In Foundations of Analysis by E. Landau (Chelsea), the arithmetic of the natural, rational, real, and complex numbers is developed soley on the basis of the five postulates of Peano. The fifth postulate is the postulate of induction.

Solutions Exercises A3-1

The solutions of several of the exercises follow the same pattern for the sequential step. In each case after assuming A_k, we add an appropriate term to each side of the equation which is the expression of A_k, and show that the resulting equation reduces to A_{k+1}. For brevity we give only the solutions of two such exercises; these will be found below in 2 and 12.

1. Prove by mathematical induction that $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$.

Follow the pattern given in 2.

5. $2n \le 2^n$

Initial Step. $2 \cdot 1 \le 2^1$

Sequential Step. Assume $A_k : 2 \cdot k \le 2^k$.

Then $2(k + 1) = 2k + 2 \le 2k + 2k,$ since $k \ge 1$.

$$\le 2 \cdot 2k \le 2 \cdot 2^k,$$

by the assumption A_k.

Therefore $2(k + 1) \le 2^{k+1}$ which is A_{k+1}

This completes the proof.

6. If $p > -1$; then, for every positive integer n,, $(1 + p)^n \ge 1 + np$.

Initial Step. $(1 + p)^1 \ge 1 + 1 \cdot p$

Sequential Step. Assume $A_k : (1 + p)^k \ge 1 + kp$.

Then, since $p > -1$, $1 + p > 0$, and we may multiply both sides of the inequality by $1 + p$ without changing its sense.

Therefore $(1 + p)^{k+1} \ge (1 + kp)(1 + p)$

$\ge 1 + kp + p + kp^2$ and dropping the positive quantity kp^2 we get $(1 + p)^{k+1} \ge 1 + (k + 1)p$ which is A_{k+1}.

This completes the proof.

7. $1 + 2 \cdot 2 + 3 \cdot 2^2 + \dots + n \cdot 2^{n-1} = 1 + (n - 1)2^n$

Follow the pattern given in 2.

Prove the following by the second principle of mathematical induction.

8. For all natural numbers n, the number $n + 1$ either is a prime or can be factored into primes.

We use the second principle of induction.

Initial Step. The number 2 is a prime.

2. By mathematical induction prove the familiar result, giving the sum of an arithmetic progression to n terms:

$$a + (a + d) + (a + 2d) + \dots + \Big(a + (n - 1)d\Big) = \frac{n}{2}[2a + (n - 1)d].$$

Initial Step. $a = \frac{1}{2}(2a + 0 \cdot d) = a$

Sequential Step. Assume $A_k : a + (a + d) + (a + 2d) + \dots + [a + (k - 1)$

$= \frac{k}{2}[2a + (k - 1)d]$.

Add $a + kd$ to both sides getting

$$\begin{aligned} a + (a + d) + \dots + [a + (k - 1)d] + (a + kd) & \\ &= \frac{k}{2}[2a + (k - 1)d] + (a + kd) \\ &= ka + \frac{k(k - 1)}{2} d + a + kd \\ &= (k + 1)a + \frac{k^2 - k + 2k}{2} d \\ &= \frac{(k + 1)}{2}(2a + kd) \\ &= \frac{k + 1}{2} (2a + [(k + 1) - 1]d) \end{aligned}$$

which is A_{k+1}.

This completes the proof.

3. By mathematical induction prove the familiar result, giving the sum of a geometric progression to n terms:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Follow the pattern given in 2.

Prove the following four statements by mathematical induction.

4. $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{1}{3}(4n^3 - n)$

Follow the pattern given in 2.

Sequential Step. Let A_n be the statement of Exercise 8, and assume that A_s is true for all natural numbers s satisfying $s \leq k$. other words, every integer $2, 3, 4, \ldots, k + 1$ is prime or a product of primes. In order to prove A_{k+1} we must show that $k + 2$ either is prime or can be factored into primes. If $k + 2$ is prime we are done. If not, we can write $k + 2$ as a product of factors r and t both less than $k + 2$, hence, both less than or equal to $k + 1$. By hypothesis, then, both factors r, t must be primes or products of primes. It follows that $k + 2$ can be written as a product of primes and that A_{k+1} is true.

9. For each natural number n greater than one, let U_n be a real number with the property that for at least one pair of natural numbers p, q with $p + q = n$, $U_n = U_p + U_q$.
When $n = 1$, we define $U_1 = a$ where a is some given real number. Prove that $U_n = na$ for all n.

Initial Step. $U_1 = 1 \cdot a$ by definition.

Sequential Step. Let A_n be the statement of Exercise 9. Using the second principle of induction we assume that for each number $s \leq k$ that A_s is true.
Now A_{k+1} must be established. But if U_{k+1} is a real number such that for at least one pair of natural numbers, f and g such that $f + g = k + 1$,

$$U_{k+1} = U_f + U_g,$$

we know that f and g must each be less than or equal to k;; and therefore U_f and U_g are real numbers to which the sequential hypothesis may be applied. Therefore

$$U_f = f \cdot a \quad \text{and} \quad U_g = g \cdot a$$

and so

$$U_{k+1} = f \cdot a + g \cdot a = (f + g) \cdot a = (k + 1)a$$

which is A_{k+1}.
This completes the proof.

10. Attempt to prove 8 and 9 from the first principle to see what difficulties arise.

In 8 note that the sequential step is based essentially upon the fact that r and t are each at most $k + 1$, not necessarily equal to $k + 1$. It would therefore be impossible to derive A_{k+1} from A_k alone and we cannot employ the first principle. Similarly in 9, we know only that f and g are at most k, not that they are necessarily equal to k. So we need to be able to refer to A_s for $s \leq k$, not to just A_k.

In the next three problems, first discover a formula for the sum, and then prove by mathematical induction that you are correct.

11. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n + 1)}$

To discover the formula for the sum, we might try writing down the sums in succession.

Thus

$$\begin{aligned} S_1 &= \frac{1}{1 \cdot 2} = \frac{1}{2} \\ S_2 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\ S_3 &= \frac{2}{3} + \frac{1}{3 \cdot 4} = \frac{3}{4} \\ S_4 &= \frac{3}{4} + \frac{1}{4 \cdot 5} = \frac{4}{5} \end{aligned}$$

So we guess $S_n = \frac{n}{n + 1}$ and try to prove it.

For the proof follow the pattern of 2.

12. $1^3 + 2^3 + 3^3 + \dots + n^3$. (Hint: Compare the sums you get here with Examples A3-1a and A3-1g in the text, or, alternatively, assume that the required result is a polynomial of degree 4.)

To guess the sum, we write down in succession the following:

$$\begin{aligned} S_1 &= 1^3 = 1 = 1^2 = 1^2; \\ S_2 &= 1 + 2^3 = 9 = 3^2 = (1 + 2)^2; \\ S_3 &= 9 + 3^3 = 36 = 6^2 = (1 + 2 + 3)^2; \\ S_4 &= 36 + 4^3 = 100 = 10^2 = (1 + 2 + 3 + 4)^2. \end{aligned}$$

We guess therefore that $S_n = (1 + 2 + 3 + \dots + n)^2$. To prove this directly by induction is quite messy (try it), but if we remember from Number 1 that $1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}$, we get a formula much easier to prove: $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$.

Now we follow the pattern of 2.

__Initial__ __Step__. $1^3 = \frac{1^2 \cdot (1 + 1)^2}{4} = 1$

__Sequential__ __Step__. Assume $A_k : 1^3 + 2^3 + \dots + k^3 = \frac{k^2(k + 1)^2}{4}$.

Add $(k + 1)^3$ to both sides, getting the following:

$$\begin{aligned} 1^3 + 2^3 + \dots + k^3 + (k + 1)^3 &= \frac{k^2(k + 1)^2}{4} + (k + 1)^3 \\ &= \frac{[(k + 1)^2(k^2 + 4(k + 1))]}{4} \\ &= \frac{(k + 1)^2(k + 2)^2}{4} \\ &= \frac{(k + 1)^2((k + 1) + 1)^2}{4}, \end{aligned}$$

which is A_{k+1}.

This completes the proof.

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1)$. (Hint: Compare this with Example A3-1g in the text.)

To guess the sum we write down in succession the following:

$$\begin{aligned} S_1 &= 1 \cdot 2 &&= 2 \\ S_2 &= 2 + 2 \cdot 3 &&= 8 \\ S_3 &= 8 + 3 \cdot 4 &&= 20 \\ S_4 &= 20 + 4 \cdot 5 &&= 40. \end{aligned}$$

This does not seem to be getting us very far. We try another approach. If you have worked Example A3-1g (and remember it) try writing S_n in this fashion,

$$
\begin{aligned}
S_n &= 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) \\
&= 1(1+1) + 2(2+1) + 3(3+1) + \ldots + n(n+1) \\
&= 1^2 + 1 + 2^2 + 2 + 3^2 + 3 + \ldots + n^2 + n \\
&= (1^2 + 2^2 + 3^2 + \ldots + n^2) + (1 + 2 + 3 + \ldots + n) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1+3)}{6} \\
&= \frac{n(n+1)(n+2)}{3} .
\end{aligned}
$$

Another way of guessing this formula would be to assume, as in Example A3-1g, that since the general term in S_n is quadratic, the formula might be cubic

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = an^3 + bn^2 + cn + d$$

and then let n take on the successive values 1, 2, 3, and 4 to determine a, b, c, and d. Thus, by successive subtractions,

$$
\left.\begin{array}{r}
a + b + c + d = 2 \\
8a + 4b + 2c + d = 8 \\
27a + 9b + 3c + d = 20 \\
64a + 16b + 4c + d = 40
\end{array}\right\}
\quad
\left.\begin{array}{r}
7a + 3b + c = 6 \\
19a + 5b + c = 12 \\
37a + 7b + c = 20
\end{array}\right\}
\quad
\left.\begin{array}{r}
12a + 2b = 6 \\
18a + 2b = 8
\end{array}\right\}
\quad 6a = 2.
$$

Therefore

$$a = \frac{1}{3}, \quad b = 1, \quad c = \frac{2}{3}, \quad d = 0,$$

and

$$S_n = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n = \frac{n(n+1)(n+2)}{3} .$$

The proof of these results follows the pattern of 2 and 12.

14. Prove for all positive integers n,

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \ldots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2 .$$

Check the initial step.

Assume A_k, and multiply both sides of the resulting equation by the appropriate factor, and reduce to get A_{k+1}.

15. Prove that $(1 + x)(1 + x^2)(1 + x^4) \ldots (1 + x^{2^n}) = \frac{1 - x^{2^{n+1}}}{1 - x}$.

Follow the pattern of Solution 14.

16. Prove that $n(n^2 + 5)$ is divisible by 6 for all integral n.

Initial Step. $1(1 + 5) = 6$ and this is divisible by 6.
Assume $A_k : k(k^2 + 5) = 6p$ where p is a positive integer.
Consider:

$$\begin{aligned}(k + 1)\left((k + 1)^2 + 5\right) &= (k + 1)^3 + 5(k + 1)\\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5\\ &= (k^3 + 5k) + (3k^2 + 3k) + 1 + 5\\ &= k(k^2 + 5) + 3k(k + 1) + 6.\end{aligned}$$

By A_k we know that $k(k^2 + 5) = 6p$, and since k is a positive integer either k or k + 1 is an even integer. Therefore the second term is divisible both by 2 and by 3, and therefore by 6. Finally we get

$$\begin{aligned}(k + 1)\left((k + 1)^2 + 5\right) &= 6p + 6q + 6\\ &= 6(p + q + 1)\end{aligned}$$

and this finishes the proof, since we know that the sum of three positive integers is a positive integer.

17. Any infinite straight line separates the plane into two parts; two intersecting straight lines separate the plane into four parts; and three non-concurrent lines, of which no two are parallel, separate the plane into seven parts. Determine the number of parts into which the plane is separated by n straight lines of which no three meet in a single common point and no two are parallel; then prove your result. Can you obtain a more general result when parallelism is permitted? If concurrence is permitted? If both are permitted?

Both our method of guessing the answer and our proof will be sequential. Let R_n be the number of regions into which the plane is divided by n lines of which no two are parallel and no three are concurrent. If we draw an $(n + 1)$-th line under the same conditions, it must meet all the other lines in n new points of intersection. In crossing n lines it must go through n + 1 regions of the plane, dividing each region into two parts, thus adding n + 1 new regions. We conclude that

$$R_{n+1} = (n + 1) + R_n.$$

Since $R_1 = 2$, this is a recursive definition for R_n. We have, plainly,

$$R_n = 2 + 2 + 3 + 4 + \dots + n = \tfrac{1}{2}(n^2 + n + 2)$$

and this result can be obtained directly from the recursion formula by a straightforward induction.

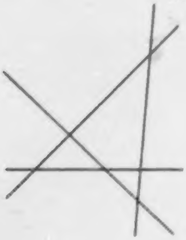
If parallelism is permitted, each pair of parallel lines existing reduces R_n by 1, since one crossing is eliminated. Thus if p lines are parallel, you can pick $\frac{p(p - 1)}{2}$ pairs of parallel lines and there will be this many fewer regions

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(p - 1)p}{2}.$$

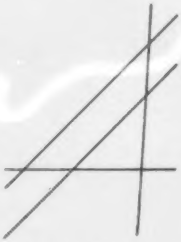
For example if four lines are drawn, three of which are parallel, there will be

$$1 + \frac{4(5)}{2} - \frac{3(2)}{2} = 8 \text{ regions.}$$

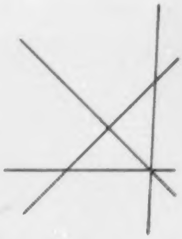
Similarly, any line which concurs with an already existing intersection point reduces the total number of intersection points by one, and the number of regions of the plane by one. Again we must remember, as in the parallel case, that pairs of extra concurrencies must all be counted. Thus if c lines concur at one point



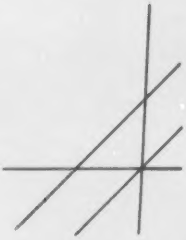
$n=4$, $p=0$, $c=0$
$R_4=11$



$n=4$, $p=2$, $c=0$
$R_4=10$



$n=4$, $p=0$, $c=3$
$R_4=10$



$n=4$, $p=2$, $c=3$
$R_4=9$

$$R_n = 1 + \frac{n(n + 1)}{2} - \frac{(c - 1)(c - 2)}{2}.$$

If a line provides both a case of parallelism and a case of concurrence, it must be counted each way in reducing the number of regions, as is shown in the figure. In general if there are j families of parallel lines with $p_1, p_2, \dots, p_j$ lines in each family and k families of concurrent lines with $c_1, c_2, \dots, c_k$ lines in each family, we have

$$\begin{aligned} R_n = 1 + \frac{n(n+1)}{2} \\ - \left[\frac{p_1(p_1-1)}{2} + \frac{p_2(p_2-1)}{2} + \dots + \frac{p_j(p_j-1)}{2}\right] \\ - \left[\frac{(c_1-1)(c_1-2)}{2} + \frac{(c_2-1)(c_2-2)}{2} + \dots + \frac{(c_k-1)(c_k-2)}{2}\right] \end{aligned}$$

The proof of this is too lengthy for insertion here.

18. Consider the sequence of fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots, \frac{p_n}{q_n}, \dots$$

where each fraction is obtained from the preceding by the rule

$$p_n = p_{n-1} + 2q_{n-1}$$

$$q_n = p_{n-1} + q_{n-1}.$$

Show that for n sufficiently large, the difference between $\frac{p_n}{q_n}$ and $\sqrt{2}$ can be made as small as desired. Show also that the approximation to $\sqrt{2}$ is improved at each successive stage of the sequence and that the error alternates in sign. Prove also that p_n and q_n are relatively prime, that is, the fraction $\frac{p_n}{q_n}$ is in lowest terms.

Let the error at the n-th stage be denoted by $e_n = \frac{p_n}{q_n} - \sqrt{2}$. We may define the error e_{n+1} at the next stage recursively in terms of e_n as follows:

$$e_{n+1} = \frac{p_n + 2q_n}{p_n + q_n} - \sqrt{2}$$

$$= \frac{\frac{p_n}{q_n} + 2}{\frac{p_n}{q_n} + 1} - \sqrt{2}$$

$$= \frac{e_n + 2 + \sqrt{2}}{e_n + 1 + \sqrt{2}} - \sqrt{2}$$

$$= \frac{e_n(1 - \sqrt{2})}{e_n + 1 + \sqrt{2}} .$$

Since $1 - \sqrt{2}$ is negative, it follows that e_{n+1} has the opposite sign from e_n, and the sign alternates if the denominator is shown to be positive. We shall prove by induction that $|e_n| < \frac{1}{2^n}$ and thereby show simultaneously that the denominator above is positive, and that the error can be made as small as desired by taking n sufficiently large.

Initial Step. $|e_1| = |1 - \sqrt{2}| = .414 \ldots < \frac{1}{2}$

Sequential Step. Assume $|e_k| < \frac{1}{2^n}$. For the denominator of e_{k+1}, we have

$$e_k + 1 + \sqrt{2} > -\frac{1}{2^k} + 1 + \sqrt{2} > -\frac{1}{2} + 1 + \sqrt{2} > \frac{1}{2} + \sqrt{2} > 1.$$

We also have $$\sqrt{2} - 1 < \frac{1}{2} .$$

It follows from the recursive expression for e_{k+1} that

$$|e_{k+1}| < \frac{1}{2}|e_k| < \frac{1}{2} \cdot \frac{1}{2^k} = \frac{1}{2^{k+1}} .$$

To prove that p_n and q_n have no common factor other than 1, we note that

$$p_n = p_{n+1} - q_{n+1}, \quad q_n = 2q_{n+1} - p_{n+1}.$$

We then reason inductively as follows:

Initial Step. The only common factor of p_1 and of q_1 is 1.

Sequential Step. Assume p_k and q_k have no common factor other than 1. If p_{k+1} and q_{k+1} had such a common factor, then, by the above formula it would have to be a common factor of p_k and q_k. Contradiction.

19. Let p be any polynomial of degree m. Let $q(n)$ denote the sum

$$q(n) = p(1) + p(2) + p(3) + \dots + p(n). \tag{1}$$

Prove that there is a polynomial q of degree $m + 1$ satisfying (1).

Initial Step. We observe that if p has degree 0, then $p = c$ where c is a constant and we have

$$p(1) + p(2) + \dots + p(n) = c + c + c + \dots + c = cn. \tag{1}$$

Hence $q(x) = cx$ is a polynomial of first degree satisfying the condition.

Sequential Step. We assume that the theorem is true for any polynomial p of degree less than or equal to k. Let

$$p(x) = ax^{k+1} + p_1(x), \qquad (a \neq 0) \tag{2}$$

where the degree of p_1 is $\leq k$..
Next we observe that

$$(x + 1)^{k+2} = x^{k+2} + (k + 2)x^{k+1} + p_2(x) \tag{3}$$

where the degree of p_2 is $\leq k$. This fact has to be proved by induction, unless the binomial theorem is taken for granted. It will be proved afterward. Setting

$$(x + 1)^{k+2} - x^{k+2} = (k + 2)x^{k+1} + p_2(x)$$

and solving for x^{k+1} we obtain in (2)

$$p(x) = \frac{a}{k + 2}[(x + 1)^{k+2} - x^{k+2}] + p_3(x) \tag{4}$$

where $$p_3(x) = p_1(x) - \frac{a}{k + 2} p_2(x)$$

and therefore the degree of $p_3(x)$ is $\leq k$.
Consequently,

$$\begin{aligned} p(1) + p(2) + \dots + p(n) = \frac{a}{k + 2} \{(2^{k+2} - 1^{k+2}) \\ + (3^{k+2} - 2^{k+2}) + \dots + [(n + 1)^{k+2} - n^{k+2}]\} \\ + p_3(1) + p_3(2) + \dots + p_3(n). \end{aligned}$$

By the induction hypothesis, there exists a $q_1(x)$ of degree $\leq k + 1$, such that

$$q_1(n) = p_3(1) + \dots + p_3(n).$$

Furthermore the expression in braces reduces by successive additions and subtractions to $(n + 1)^{k+2} - 1^{k+2}$, and we obtain the desired polynomial,

$$q(x) = \frac{a}{k + 2}[(x + 1)^{k+2} - 1^{k+2}] + q_1(x)$$

where $\quad q(n) = p(1) + \dots + p(n)$.

Now we prove (3):

Initial Step. If $k = 0$, $(x + 1)^2 = x^2 + (0 + 2)x + 1$ and the degree of 1 is 0.

Sequential Step.
$$\begin{aligned}(x+1)^{k+3} &= x^{k+2}(x+1) + (k+2)x^{k+1}(x+1) + (x+1)p_2(x)\\ &= x^{k+3} + (k+3)x^{k+2} + [(k+2)x^{k+1} + (x+1)p_2(x)]\end{aligned}$$

20. Let the function $f(n)$ be defined recursively as follows:
Initial Step. $f(1) = 3$
Sequential Step. $f(n + 1) = 3^{f(n)}$
In particular, we have $f(3) = 3^{3^3} = 3^{27}$, etc.
Similarly, $g(n)$ is defined by
Initial Step. $g(1) = 9$
Sequential Step. $g(n + 1) = 9^{g(n)}$
Find the minimum value m for each n such that $f(m) \geq g(n)$.

It is easily seen that $g(n) > f(n)$ for all n. We shall prove that $f(n + 1) > g(n)$ for all n and, hence, that $m = n + 1$ is the least value for which $f(m) > g(n)$.

Initial Step. If $n = 1$, $f(2) = 3^3 = 27$, and $g(1) = 9$. Consequently $f(2) > g(1)$. More strongly, $f(2) > 2g(1) + 1$; and we shall prove generally $f(n + 1) > 2g(n) + 1$.

Sequential Step. Suppose:

$$f(k + 1) > 2g(k) + 1 > g(k).$$

Then,

$$\begin{aligned}f(k + 2) = 3^{f(k+1)} &> 3^{2g(k)+1} \geq 3 \cdot 3^{2g(k)}\\ &\geq 3 \cdot 9^{g(k)} \geq 3g(k + 1)\\ &> 2g(k + 1) + 1\\ &> g(k + 1).\end{aligned}$$

21. Prove for all natural numbers n, that $\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$ is an integer. (Hint: Try to express $x^n - y^n$ in terms of $x^{n-1} - y^{n-1}$, $x^{n-2} - y^{n-2}$, etc.)

Let $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$. We will use the second principle.

Initial Step. $F_1 = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = 1$

$$F_2 = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

Sequential Step. Assume F_s is an integer for all $s \leq k$.

Consider $F_{k+1} = \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$. For brevity we

write $1+\sqrt{5} = x$ and $1-\sqrt{5} = y$.

Then

$$F_{k+1} = \frac{x^{k+1} - y^{k+1}}{2^{k+1}\sqrt{5}} = \frac{x^{k+1} + x^k y - x^k y + xy^k - xy^k - y^{k+1}}{2^{k+1}\sqrt{5}}$$

$$= \frac{x^k(x+y) - xy(x^{k-1} - y^{k-1}) - y^k(x+y)}{2^{k+1}\sqrt{5}}$$

$$= \frac{(x+y)(x^k - y^k)}{2^{k+1}\sqrt{5}} - \frac{xy(x^{k-1} - y^{k-1})}{2^{k+1}\sqrt{5}}$$

$$= \frac{x+y}{2}\left(\frac{x^k - y^k}{2^k\sqrt{5}}\right) - \frac{xy}{4}\left(\frac{x^{k-1} - y^{k-1}}{2^{k-1}\sqrt{5}}\right)$$

$$= \frac{(1+\sqrt{5}) + (1-\sqrt{5})}{2}F_k - \frac{(1+\sqrt{5})(1-\sqrt{5})}{4}F_{k-1}$$

$$= F_k + F_{k-1}.$$

but by the assumption of the sequential step we know F_k and F_{k-1} are integers. Therefore F_{k+1} is an integer. This completes the theorem.

Solutions Exercises A3-2a

1. Prove

$$\sum_{k=1}^{n} (\alpha f_k + \beta g_k) = \alpha \sum_{k=1}^{n} f_k + \beta \sum_{k=1}^{n} g_k$$

The linearity of summation is a consequence of the additive and multiplicative properties of real numbers and follows easily by mathematical induction.

2. Write each of the following sums in expanded form and evaluate:

(a) $\sum_{k=1}^{5} 2k$ $\qquad 2 + 4 + 6 + 8 + 10 = 30$

(b) $\sum_{j=2}^{6} j^2$ $\qquad 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 90$

(c) $\sum_{r=-1}^{3} (r^2 + r - 12)$ $\qquad (-12) + (-12) + (-10) + (-6) + (0) = -40$

(d) $\sum_{m=2}^{5} m(m-1)(m-2)$ $\qquad 0 + 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 + 5 \cdot 4 \cdot 3 = 90$

(e) $\sum_{i=0}^{10} 2^i$ $\qquad 1 + 2 + 2^2 + 2^3 + \dots + 2^{10} = 2^{11} - 1 = 2043$

(f) $\sum_{r=0}^{4} \frac{4!}{r!(4-r)!}$ $\qquad 1 + 4 + 6 + 4 + 1 = 16$

3. Which of the following statements are true and which are false? Justify your conclusions.

(a) $\sum_{j=3}^{10} 4 = 7 \cdot 4 = 28$

(b) $\sum_{j=m}^{n} 4 \; \text{[illegible]} \; (n - m + 1)$

(c) $\sum_{k=1}^{10} k^2 = 10 \sum_{k=1}^{9} k^2$

(d) $\sum_{k=1}^{1000} k^2 = 5 + \sum_{k=3}^{1000} k^2$

(e) $\sum_{k=1}^{n} k^3 = n^3 + \sum_{j=2}^{n} (j - 1)^3$

(f) $\sum_{m=1}^{10} k^2 = \left(\sum_{m=1}^{10} k \right)^2$

(g) $\sum_{m=1}^{10} k^3 = \left(\sum_{m=1}^{10} k \right)^2$

(h) $\sum_{i=0}^{n} i(i - 1)(n - i) = \sum_{i=2}^{n-1} i(i - 1)(n - i)$

(i) $\sum_{k=0}^{m} f(a_{m-k}) = \sum_{k=0}^{m} f(a_k)$.

(j) $n \sum_{k=0}^{n} A_k - \sum_{k=0}^{n} kA_k = \sum_{k=0}^{n} kA_{n-k}$

(k) $\sum_{k=0}^{m} k^2(A_k - A_{m-k}) = m^2 \sum_{k=0}^{m} A_{m-k} - 2m \sum_{k=0}^{m} k \, A_{m-k}$

(a) False; $\sum_{j=3}^{10} 4 = 8 \cdot 4 = 32$

(b) True

(c) False; $\sum_{k=1}^{10} k^2 = 10^2 + \sum_{k=1}^{9} k^2 < 10 \sum_{k=1}^{9} k^2$

(d) True

(e) True

(f) False; unless $k = 0$.

(g) False; unless $k = 0$.

(h) True; the missing terms are zero.

(i) True; $m - k$ takes on the same values as k.

(j) True: $\sum_{k=0}^{n} k\, A_{n-k} = \sum_{k=0}^{n} (n - k)A_k$ by (i).

(k) True; follows by applying (i) to $\sum_{k=0}^{m} (m^2 - 2mk + k^2)A_{m-k}$.

4. Evaluate $\sum_{k=1}^{n} f(\frac{k}{n})(\frac{b - a}{n})$ if $f(x) = x^2$, $a = 0$, $b = 1$ and

(a) $n = 2$ $\quad \frac{5}{8}$

(b) $n = 4$ $\quad \frac{15}{32}$

(c) $n = 8$ $\quad \frac{102}{256}$

5. Subdivide the interval $[0,1]$ into n equal parts. In each subinterval obtain upper and lower bounds for x^2. Using sigma notation, use these upper and lower bounds to obtain expressions for upper and lower estimates of the area under the curve $y = x^2$ on $[0,1]$. If you can evaluate these sums without reading elsewhere, do so.

$$\text{Lower sum} = \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \frac{1}{n^3}\sum_{k=0}^{n-1} k^2 = \frac{(n-1)(n)(2n-1)}{6n^3} .$$

$$\text{Upper sum} = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2 = \text{lower sum} + \frac{1}{n} .$$

6. (a) Write out the sum of the first 7 terms of an arithmetic progression with first term a and common difference d. Express the same sum in sigma notation.

$$(a) + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + (a + 5d) + (a + 6d)$$

$$= \sum_{n=1}^{7} \{a + (n - 1)d\}.$$

(b) In sigma notation, write the expression for the sum of the first n terms of a geometric progression with first term a and common ratio r.

$$\sum_{k=1}^{n} ar^{k-1}$$

7. (a) Consider a function f defined by

$$f(n) = \sum_{r=1}^{n} \{(r - 1)(r - 2)(r - 3)(r - 4)(r - 5) + r\}.$$

Find $f(n)$ for $n = 1, 2, \ldots, 5$.

$$f(n) = \frac{n(n + 1)}{2} \quad \text{for } n = 1, 2, \ldots, 5.$$

(b) Give an example of a function g (similar to that in (a)) such that

$$g(n) = 1,\ n = 1, 2, \ldots, 10^6,$$
$$g(10^6 + 1) = 0.$$

$$g(n) = 1 - \frac{(n - 1)(n - 2)(n - 3) \ldots (n - 10^6)}{10^6!}$$

8. Write each of the following sums in expanded form and evaluate:

(a) $$\sum_{n=1}^{4} \left\{\sum_{r=1}^{3} r(n - r)\right\}$$

$$\sum_{n=1}^{4} \{1(n - 1) + 2(n - 2) + 3(n - 3)\} = \sum_{n=1}^{4} \{6n - 14\} = 4$$

(b) $$\sum_{n=1}^{N} \left\{\sum_{r=1}^{R} (rn - 1)\right\}$$

$$\sum_{n=1}^{N} \{(n - 1) + (2n - 1) + \ldots + (Rn - 1)\} = \sum_{n=1}^{N} \frac{n(R)(R + 1)}{2} - R$$

$$= \frac{N(N + 1)(R)(R + 1)}{4} - RN$$

9. The double sum $\sum_{i=0}^{m} \sum_{j=0}^{n} F(i,j)$ is a shorthand notation for

$$\sum_{i=0}^{m} \{F(i,0) + F(i,1) + \ldots + F(i,n)\} \quad \text{or} \quad \begin{aligned} &F(0,0) + F(0,1) + \ldots + F(0,n) \\ &+ F(1,0) + F(1,1) + \ldots + F(1,n) \\ &\quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &+ F(m,0) + F(m,1) + \ldots + F(m,n). \end{aligned}$$

In particular,

$$\sum_{i=1}^{2}\sum_{j=1}^{3} i \cdot j = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 = 18$$

Evaluate:

(a) $\sum_{i=1}^{m}\sum_{j=1}^{n} i \cdot j$

(b) $\sum_{i=1}^{m}\sum_{j=1}^{n} (i + j)$

(c) $\sum_{i=1}^{m}\sum_{j=1}^{n} \max\{i,j\}$

(d) $\sum_{i=1}^{m}\sum_{j=1}^{n} \min\{i,j\}$

(a) $\sum_{i=1}^{m} i \left\{\sum_{j=1}^{n} j\right\} = \frac{m(m + 1)(n)(n + 1)}{4}$

(b) $\sum_{i=1}^{m} \{ni + \frac{n(n + 1)}{2}\} = \frac{n(m)(m + 1)}{2} + \frac{mn(n + 1)}{2} = \frac{mn(m + n + 2)}{2}$.

(c) If $n \geq m$, we have

$$\sum_{i=1}^{m} \left\{\sum_{j=1}^{i} i + \sum_{j=i+1}^{n} j\right\} = \sum_{i=1}^{m} \{i^2 + \frac{n(n + 1)}{2} - \frac{i(i + 1)}{2}\}$$

$$= \frac{1}{2}\sum_{i=1}^{m} i(i - 1) + \frac{1}{2}\sum_{i=1}^{m} n(n + 1)$$

$$= \frac{(m - 1)m(m + 1)}{6} + \frac{mn(n + 1)}{2}$$

For $m \geq n$, just interchange m and n (by symmetry).

(d) This can also be done similarly, i.e.,

$$\sum_{i=1}^{m} \left\{\sum_{j=1}^{i} j + \sum_{j=i+1}^{n} i\right\}$$

Alternatively,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \{\max(i,j) + \min(i,j)\} = \sum_{i=1}^{m} \sum_{j=1}^{n} (i + j)$$

Now use (b) and (c), to give

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \min(i,j) = \frac{m^2 n}{2} - \frac{(m - 1)(m)(m + 1)}{6} .$$

10. (a) Show that $\frac{1}{k(k - 1)} = \frac{1}{k - 1} - \frac{1}{k}$, $k \neq 0, 1$.

$$\frac{1}{k - 1} - \frac{1}{k} = \frac{k - (k - 1)}{(k - 1)k} = \frac{1}{k(k - 1)} , \ k \neq 0.$$

(b) Evaluate $\sum_{k=2}^{1000} \frac{1}{k(k - 1)}$.

$$\sum_{k=2}^{1000} \frac{1}{k(k - 1)} = \sum_{k=2}^{1000} \left(\frac{1}{k - 1} - \frac{1}{k}\right) = \frac{1}{2 - 1} - \frac{1}{1000} = \frac{999}{1000} .$$

In general, $\sum_{k=2}^{n} \frac{1}{k(k - 1)} = 1 - \frac{1}{n}$.

11. If $S(n) = \sum_{i=1}^{n} f(i)$ determine $f(m)$ in terms of the sum function S.

$f(1) = S(1)$

$$S(m) - S(m - 1) = \sum_{i=1}^{m} f(i) - \sum_{i=1}^{m-1} f(i) = f(m), \ m > 1.$$

12. Determine $f(m)$ in the following summation formulae: (See Number 11.)

(a) $1 = \sum_{i=1}^{n} f(i)$ $\qquad$ $f(1) = 1, \ f(m) = 0, \ m > 1$.

(b) $n = \sum_{i=1}^{n} f(i)$ $\qquad f(m) = 1,\ m \geq 1.$

(c) $n^2 = \sum_{i=1}^{n} f(i)$ $\qquad f(m) = m^2 - (m - 1)^2 = 2m - 1,\ m \geq 1.$

(d) $an^2 + bn + c = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= a + b + c, \\ f(m) &= am^2 + bm + c - a(m-1)^2 - b(m-1) - c \\ &= a(2m - 1) + b,\ m > 1. \end{aligned}$$

(e) $\cos nx = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= \cos x, \\ f(m) &= \cos mx - \cos(m - 1)x \\ &= -2 \sin \frac{x}{2} \sin(m - \frac{1}{2})x,\ m > 1. \end{aligned}$$

(f) $\sin(an + b) = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= \sin(a + b), \\ f(m) &= \sin(am + b) - \sin\Big(a(m - 1) + b\Big) \\ &= 2 \sin \frac{a}{2} \cos\Big(am + b - \frac{a}{2}\Big),\ m > 1. \end{aligned}$$

(g) $n! = \sum_{i=1}^{n} f(i)$

$$\begin{aligned} f(1) &= 1, \\ f(m) &= m! - (m - 1)! \\ &= (m - 1)!(m - 1),\ m > 1. \end{aligned}$$

13. Binomial Theorem:

We define $\binom{n}{r} = \frac{n!}{(n - r)!r!}$ where r, n are integers such that $0 \leq r \leq n$. Also $0! = 1$ and $\binom{n}{r} = 0$ if $r > n$. Show that

(a) $\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{1} = \binom{n}{n-1} = n$

$$\binom{n}{0} = \frac{n!}{n!0!} = \frac{n!}{n! \cdot 1} = 1$$

$$\binom{n}{n} = \frac{n!}{0!n!} = 1$$

$$\binom{n}{1} = \frac{n!}{(n-1)!1!} = \frac{n!}{(n-1)!(n-(n-1))!}$$

$$= \frac{n!}{(n-1)!} = n$$

(b) $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \binom{n}{n-r}$$

$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

$$\binom{n}{r} + \binom{n}{r+1}$$

$$= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r-1)!(r+1)!}$$

$$= \frac{n!(r+1)}{(n-r)!(r+1)!} + \frac{n!(n-r)}{(n-r)!(r+1)!}$$

$$= \frac{n!(n+1)}{(n-r)!(r+1)!} = \frac{(n+1)!}{(n-r)!(r+1)!}$$

$$= \binom{n+1}{r+1}$$

(c) Establish the Binomial Theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + nx^{n-1} y + \dots + nxy^{n-1} + y^n, \quad n = 0, 1, 2, \dots$$

by mathematical induction.

For $n = 1$; $(x + y) = \binom{1}{0}xy^0 + \binom{1}{1}x^0y = x + y$.

Now assume $(x+y)^k = \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^r$ for all $k \leq n$.

We show this implies the truth of the theorem for $k = n + 1$.

$$(x + y)^{n+1} = (x + y)^n(x + y) = \left(\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r\right)(x + y)$$

$$= \sum_{r=0}^{n}\left(\binom{n}{r} x^{n-r+1} y^r + \binom{n}{r} x^{n-r} y^{r+1}\right)$$

$$= \sum_{r=1}^{n}\left(\left(\binom{n}{r} + \binom{n}{r-1}\right) x^{n-r+1} y^r\right) + \binom{n}{0} x^{n+1} y^0 + \binom{n}{n} x^0 y^{n+1}$$

(using (b))

$$= \sum_{r=1}^{n} \binom{n+1}{r} x^{n-r+1} y^r + \binom{n+1}{0} x^{n+1} y^0 + \binom{n+1}{n+1} x^0 y^{n+1}$$

$$= \sum_{r=0}^{n+1} \binom{n+1}{r} x^{n+1-r} y^r$$

$$= \sum_{r=0}^{k} \binom{k}{r} x^{k-r} y^k, \text{ where } k = n + 1.$$

14. Using the binomial theorem, give the expansions for the following:

(a) $(x + y)^3$

$$(x + y)^3 = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

(b) $(x - y)^3$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - 3y^3$$

(c) $(2x - 3y)^3$

$$(2x - 3y)^3 = \binom{3}{3}(2x)^3 + \binom{3}{2}(2x)^2(-3y)$$
$$+ \binom{3}{1}(2x)(-3y)^2 + \binom{3}{0}(-3y)^3$$
$$= 8x^3 - 36x^2y + 54xy^2 - 23y^3$$

(d) $(x - 2y)^5$

$$(x - 2y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4(-2y) + \binom{5}{3}x^3(-2y)^2$$
$$+ \binom{5}{2}x^2(-2y)^3 + \binom{5}{1}x(-2y)^4 + \binom{5}{0}(-2y)^5$$
$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

15. Evaluate the following sums.

(a) $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$

Since $(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$, $2^n = \sum_{r=0}^{n} \binom{n}{r}$ by setting $x = 1$.

(b) $\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n \binom{n}{n} = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$

If instead, we let $x = -1$, $0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r}$.

16. Sum $\sum_{r=0}^{n} r\binom{n}{r}$ by first showing $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$ and using 15(a).

By 13(a), $\sum_{r=0}^{n} r\binom{n}{r} = \sum_{r=0}^{n} (n - r)\binom{n}{n-r} = \sum_{r=0}^{n} (n - r)\binom{n}{r}$..

Thus, $2\sum_{r=0}^{n} r\binom{n}{r} = n\sum_{r=0}^{n} \binom{n}{r} = n \cdot 2^n$ by 15(a), and the sum is $n \cdot 2^{n-1}$...

17. If $P_n(x)$ denotes a polynomial of degree n such that $P_n(x) = 2^x$ for $x = 0, 1, 2, \dots, n$ find $P_n(n + 1)$.

For this problem, it will be convenient to set

$$Q_r(x) = \frac{x(x - 1) \dots (x - r + 1)}{r!}$$

where r is a non-negative integer. Note for any integer $n \geq r$ that $Q_r(n) = \binom{n}{r}$. Consider the n-th degree polynomial

$$P_n(x) = \sum_{r=0}^{n} Q_r(x) = 1 + x + \frac{x(x-1)}{1 \cdot 2} + \dots + \frac{x(x-1)\dots(x-r+1)}{r!}$$

$P_n(0) = \binom{0}{0} = 1,$

$P_n(1) = \binom{1}{0} + \binom{1}{1} = 2,$

$P_n(2) = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 2^2,$

$P_n(x) = 2^x$ for $x = 0, 1, 2, \dots, n$ by Number 15(a), and thus satisfies our requirements.

$$\begin{aligned} P_n(n+1) &= \binom{n+1}{0} + \binom{n+1}{1} + \dots + \binom{n+1}{n} \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} - \binom{n+1}{n+1} \\ &= 2^{n+1} - 1. \end{aligned}$$

Solutions Exercises A3-2b

1. Write the following sums in telescoping form, i.e., in the form $\sum_{k=1}^{n} \{u(k) - u(k-1)\}$, and evaluate

   (a) $\sum_{k=1}^{n} k(k+1)$

   (b) $\sum_{k=1}^{n} k(2k-1)$

   (c) $\sum_{k=1}^{n} 2k(2k+1)$

   (d) $\sum_{k=1}^{n} k(k+1)(k+2)$

   (e) $\sum_{k=1}^{n} k^3$

   (f) $\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}$

   (g) $\sum_{k=1}^{n} k \cdot k!$

   (h) $\sum_{k=1}^{n} r^k$

(a) $\frac{1}{3}\sum_{k=1}^{n}\{k(k + 1)(k + 2) - (k - 1)k(k + 1)\} = \frac{n(n + 1)(n + 2)}{3}$

(b) $k(2k - 1) = 2k(k + 1) - 3k$. Using (a) and $3k = \frac{3}{2}\{k(k+1) - (k-1)k\}$, the sum is $\frac{2n(n + 1)(n + 2)}{3} - \frac{3n(n + 1)}{2}$.

(c) $2k(2k + 1) = 4k(k + 1) - 2k$. Using (a) and (b), the sum is $\frac{4n(n + 1)(n + 2)}{3} - n(n + 1)$.

(d) Here, $u(k) = \frac{k(k + 1)(k + 2)(k + 3)}{4}$ and the sum is

$\frac{n(n + 1)(n + 2)(n + 3)}{4}$.

(e) $k^3 = k(k + 1)(k + 2) - 3k(k + 1) - k$. Whence

$u(k) = \frac{k(k + 1)(k + 2)(k + 3)}{4} - \frac{3k(k + 1)(k + 2)}{3} - \frac{k(k + 1)}{2}$

and the sum is

$\frac{n(n + 1)(n + 2)(n + 3)}{4} - n(n + 1)(n + 2) - \frac{n(n + 1)}{2}$.

(f) Here, $u(k) = -\frac{1}{2(k + 1)(k + 2)}$ and the sum is

$\frac{1}{2}\{\frac{1}{1 \cdot 2} - \frac{1}{(n + 1)(n + 2)}\}$.

(g) Here, $u(k) = (k + 1)!$ and the sum is $(n + 1)! - 1$.

(h) Here, $u(k) = \frac{r^{k+1}}{r - 1}$ and the sum is $\frac{r^{n+1} - 1}{r - 1}$, $r \neq 1$.

2. Using $\sum_{k=1}^{n}\{u(k) - u(k - 1)\} = u(n) - u(0)$, establish a short dictionary of summation formulae by considering the following functions u:

(a) $(a + kd)(a + (k + 1)d) \ldots (a + (k + p)d)$

(b) The reciprocal of (a).

(c) r^k

(d) kr^k

(e) k^2r^k

(f) $k!$

(g) $(k!)^2$

(h) $\arctan k$

(i) $k \sin k$

(a) $$(p + 1)d \sum_{k=1}^{n} (a + kd)(a + (k + 1)d) \ldots (a + (k + p - 1)d)$$

$$= (a + nd)(a + (n + 1)d)\ldots(a + (n + p)d) - a(a + d)\ldots(a + pd).$$

(b) $$(p + 1)d \sum_{k=1}^{n} \{(a + (k - 1)d)(a + kd)\ldots(a + (k + p)d)\}^{-1}$$

$$= \{a(a + d)\ldots(a + pd)\}^{-1} - \{(a + nd)(a + (n + 1)d)\ldots(a + (k + p)d)\}^{-1}.$$

(c) $$(r - 1) \sum_{k=1}^{n} r^{k-1} = r^{n} - 1.$$

(d) $$(r - 1) \sum_{k=1}^{n} kr^{k-1} + \sum_{k=1}^{n} r^{k-1} = nr^{n} \quad \text{or}$$

$$\sum_{k=1}^{n} kr^{k-1} = \frac{n(r - 1)r^{n} - r^{n} + 1}{(r - 1)^{2}}.$$

(e) $$(r - 1) \sum_{k=1}^{n} k^{2}r^{k-1} = n^{2}r^{n} + \sum_{k=1}^{n} r^{k-1} - 2 \sum kr^{k-1}.$$

(Now use (c) and (d).)

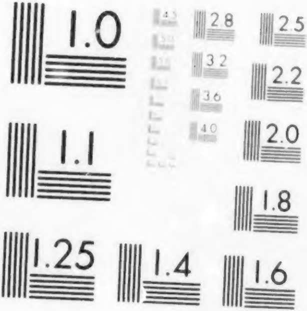
(f) $$\sum_{k=1}^{n} (k - 1)!(k - 1) = n! - 1.$$

(g) $$\sum_{k=1}^{n} (k^{2} - 1)[(k - 1)!]^{2} = (n!)^{2} - 1$$

(h) $$\sum_{k=1}^{n} \arctan \frac{1}{k^{2} - k + 1} = \arctan n$$

(i) $$\tfrac{1}{2}(\sin \tfrac{1}{2}) \sum_{k=1}^{n} k \cos(k - \tfrac{1}{2}) = n \sin n - \sum_{k=1}^{n} \sin(k - 1)$$

(Now use Equation 8.)


1.0
4.5
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5.6
6.3
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3.6
4.0
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2.2
2.0
1.1
1.8
1.25
1.4
1.6

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3. Simplify:

$$\frac{\sin x + \sin 3x + \dots + \sin((2n - 1)x)}{\cos x + \cos 3x + \dots + \cos((2n - 1)x)}.$$

Since

$$\sum_{k=1}^{n} \cos(ak + b - \tfrac{a}{2}) = \cos(b + \tfrac{an}{2}) \frac{\sin \frac{an}{2}}{\sin \frac{a}{2}},$$

$$\sum_{k=1}^{n} \sin x(2k - 1) = \frac{\sin xn \sin xn}{\sin x}$$

by letting $a = -2x$, $b = \frac{\pi}{2}$, and

$$\sum_{k=1}^{n} \cos x(2k - 1) = \frac{\cos xn \sin xn}{\sin x}$$

by letting $a = 2x$, $b = 0$..

Whence,

$$\frac{\displaystyle\sum_{k=1}^{n} \sin x(2k - 1)}{\displaystyle\sum_{k=1}^{n} \cos x(2k - 1)} = \tan xn.$$

4. Another method for summing $\sum P(k)$ (P - a polynomial) can be obtained by using a special case of Number 2a, i.e.,

$$\sum_{k=1}^{n} [(k + 1)(k)(k - 1)\dots(k - r + 1) - (k)(k - 1)(k - 2)\dots(k - r)]$$

$$= (n + 1)(n)(n - 1)\dots(n - r + 1),$$

or

$$\sum_{k=1}^{n} k(k - 1)\dots(k - r + 1) = \frac{(n + 1)(n)(n - 1)\dots(n - r + 1)}{r + 1}.$$

First, we show how to represent any polynomial $P(k)$ of r-th degree in the form

$$(1) \quad P(k) = a_0 + a_1 k + \frac{a_2 k(k - 1)}{2!} + \dots + \frac{a_r k(k - 1)\dots(k - r + 1)}{r!}.$$

If $k = 0$, then $a_0 = P(0)$; if $k = 1$, then $a_1 = P(1) - P(0)$; if $k = 2$, then $a_2 = P(2) - 2P(1) + P(0)$. In general, it can be shown that

(ii) $a_m = P(m) - \binom{m}{1}P(m-1) + \binom{m}{2}P(m-2) - \dots + (-1)^m P(0),$

$$m = 0, 1, \dots, r.$$

Since both sides of (i) are polynomials of degree r and (i) is satisfied for $m = 0, 1, \dots, r$, it must be an identity.

Now sum $\sum_{k=1}^{n} P(k)$.

$$\sum_{k=1}^{n} P(k) = a_0 n + \frac{a_1(n+1)(n)}{2!} + \dots + \frac{a_r(n+1)(n)(n-1)\dots(n-r+1)}{r!} .$$

5. Using Problem 4, find the following sums:

(a) $\sum_{k=1}^{n} k^2$

$k^2 = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!}$ where $a_0 = 0^2$, $a_1 = 1^2 - 0^2 = 1$, $a_2 = 2^2 - 2(1) + 0 = 2$. Thus, $k^2 = k + k(k-1)$ and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)}{2} + \frac{(n+1)(n)(n-1)}{3} = \frac{n(n+1)(2n+1)}{6} .$$

(b) $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$

$k^3 = a_0 + a_1 k + \frac{a_2 k(k-1)}{2!} + \frac{a_3 k(k-1)(k-2)}{3!}$ where $a_0 = 0$, $a_1 = 1$, $a_2 = 2^3 - 2(1) = 6$, $a^3 = 3^3 - 3(8) + 3(1) = 6$. Thus, $k^3 = k + 3k(k-1) + k(k-1)(k-2)$ (compare with Number 1e) and

$$\sum_{k=1}^{n} k^3 = \frac{(n+1)n}{2} + \frac{3(n+1)(n)(n-1)}{3} + \frac{(n+1)(n)(n-1)(n-2)}{4}$$

$$= \frac{n^2(n+1)^2}{4} .$$

$$\sum_{k=1}^{n} k = \frac{(n+1)(n)}{2}. \quad \text{Finally,} \quad \sum_{k=1}^{n} k^3 - \left(\sum_{k=1}^{n} k\right)^2 = 0.$$

(c) $\sum_{k=1}^{n} k^4$

$$k^4 = a_0 + a_1 k + \dots + \frac{a_4 k(k-1)(k-2)(k-3)}{4!}$$

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 2^4 - 2(1) = 14,$

$a_3 = 3^4 - 3(2^4) + 3(1) = 36,$

$a_4 = 4!.$ Whence,

$$\sum_{k=1}^{n} k^4 = \frac{(n+1)(n)}{2} + \frac{7(n+1)(n)(n-1)}{3} + \frac{6(n+1)(n)(n-1)(n-2)}{4} + \frac{(n+1)(n)(n-1)(n-2)(n-3)}{5}.$$

6. (a) Establish Equation (ii) of Number 4.

Since $a_0, a_1, \dots, a_r$ are defined by the equation

$$P(k) = a_0\binom{k}{0} + a_1\binom{k}{1} + \dots + a_r\binom{k}{r},$$

we have the following r linear equations for the a_i's:

$$\begin{aligned} P(0) &= a_0\binom{0}{0}, \\ P(1) &= a_0\binom{1}{0} + a_1\binom{1}{1}, \\ P(2) &= a_0\binom{2}{0} + a_1\binom{2}{1} + a_2\binom{2}{2}, \\ &\vdots \\ P(r) &= a_0\binom{r}{0} + a_1\binom{r}{1} + \dots + a_r\binom{r}{r}. \end{aligned}$$

Our proof is by mathematical induction. Assume that

$$(A) \quad a_n = P(n)\binom{n}{0} - P(n-1)\binom{n}{1} + \dots + (-1)^n P(0)\binom{n}{n} = \sum_{k=0}^{n} (-1)^k P(n-k)\binom{n}{k}$$

is valid for $n = 0, 1, 2, \dots, m - 1$. We now wish to show that the expression for a_n is also valid for $n = m$. This is equivalent to showing

$$(B) \quad P(m) = a_0\binom{m}{0} + a_1\binom{m}{1} + \dots + a_m\binom{m}{m} = \sum_{j=0}^{m} a_j\binom{m}{j}$$

(for the values of a_n given above, $n = 1, 2, \dots, m$. This will involve manipulations on double series.

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j}(-1)^k P(j-k)\binom{j}{k}$$

(by substituting for a_j in (A)).

Now let $k = j - i$. Then,

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{j=0}^{m} \sum_{i=0}^{j} \binom{m}{j}(-1)^{j-i} P(i)\binom{j}{j-i}.$$

Noting that $\binom{j}{j-i} = \binom{j}{i}$ and interchanging order of summation, we get

$$\sum_{j=0}^{m} a_j\binom{m}{j} = \sum_{i=0}^{m} \sum_{j=i}^{m} \binom{m}{j}(-1)^{j-i} P(i)\binom{j}{i}$$

$$= \sum_{i=0}^{m} P(i) \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i}.$$

Since $\binom{m}{j}\binom{j}{i} = \binom{m}{i}\binom{m-i}{j-i}$,

$$\sum_{j=i}^{m} (-1)^{j-i}\binom{m}{j}\binom{j}{i} = \sum_{j=i}^{m} (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}.$$

Now let $j = i + r$, which reduces the last summation to

$$\binom{m}{i}\sum_{r=0}^{m-r} (-1)^r\binom{m-i}{r} = 0 \quad \text{if} \quad i \neq m$$

$$= 1 \quad \text{if} \quad i = m$$

(see Exercises A3-2a, No. 15b).

Finally, $\sum_{j=0}^{m} a_j\binom{m}{j} = P(m)$ which was to be shown.

Since our inductive hypothesis (A) is valid for $n = 0$, it is valid for all n.

(b) Show that a_m is zero for $m > r$.

Suppose we wanted the equation $F(x) = a_0\binom{x}{0} + a_1\binom{x}{C} + \dots + a_m\binom{x}{m}$ (where m is any number $> r$), to be satisfied for $x = 0, 1, 2, \dots, m$ where $F(x)$ is some given function. By setting $x = 0, 1, 2, \dots, m$ in turn the a_1's will have to satisfy

$$\begin{aligned} F(0) &= a_0\binom{0}{0}, \\ F(1) &= a_0\binom{1}{0} + a_1\binom{1}{1}, \\ &\vdots \qquad\qquad\quad \vdots \\ F(m) &= a_0\binom{m}{0} + a_1\binom{m}{1} + \dots + a_m\binom{m}{m}. \end{aligned}$$

It follows (from algebra) that this system of $(m + 1)$ linear equations in $(m + 1)$ unknowns has a unique solution for all $F(x)$. By our inductive argument in part (a), the solution is given as

$$a_n = F(n)\binom{n}{0} - F(n - 1)\binom{n}{1} + \dots + (-1)^n F(0)\binom{n}{n}$$

for $n = 0, 1, 2, \dots, m$.

If we now choose $F(x)$ to be the polynomial $P(x)$ of degree r in part (a), then $P(x)$ is identical to

$$a_0\binom{x}{0} + a_1\binom{x}{1} + \dots + a_r\binom{x}{r}$$

(from Problem 4). It then follows that

$$a_{r+1}\binom{x}{r+1} + a_{r+2}\binom{x}{r+2} + \dots + a_m\binom{x}{m}$$

vanishes for $x = 0, 1, 2, \dots, m$. If a polynomial of degree m vanishes for $m + 1$ different values it must identically vanish. Therefore,

$$a_{r+1} = a_{r+2} = \dots = a_m = 0$$

for all $m > r$.

Teachers Commentary

Appendix 4

FURTHER TECHNIQUES OF INTEGRATION

A4-1. Substitutions of Circular Functions

An integral of a rational combination of sinh x and cosh x can be transformed into an integral of a rational function by a substitution exploiting the analogy between circular and hyperbolic functions. However, it is simpler to recognize that the integrand is a rational function of e^x (See Exercises A4-1, No. 10).

The integration of $\frac{1}{\cos\theta}$ which occurs in Example A4-1d may be accomplished by the substitution $u = \sin\theta$,, as follows:

$$\int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta}{1 - \sin^2\theta}\, d\theta = \int \frac{du}{1 - u^2}$$

$$= \operatorname{argtanh} \sin\theta + C$$

$$= \frac{1}{2} \log \frac{1 + \sin\theta}{1 - \sin\theta} + C.$$

Solutions Exercises A4-1

1. Integrate the following functions, the numbers a and b being positive.

(a) $\frac{\sqrt{a^2 - x^2}}{x^2}$. Set $x = a\cos\theta$.

$$-I = -\int \tan^2\theta\, d\theta = -\int [(1 + \tan^2\theta) - 1]\, d\theta$$

$$= \theta - \tan\theta + C$$

$$= \arccos\frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{x} + C.$$

(b) $\dfrac{\sqrt{1 + x^2}}{x^4}$. Set $x = \tan t$.

$$I = \int \frac{\cos t}{\sin^4 t}\,dt = -\frac{1}{3\sin^3 t} + C$$

$$= -\frac{1}{3}\left(\frac{\sqrt{1 + x^2}}{x}\right)^3 + C.$$

(Alternatively, set $u = \dfrac{1}{x^2}$ to obtain $I = -\dfrac{1}{2}\displaystyle\int \sqrt{1 + u}\,du$.)

(c) $x^2\sqrt{a^2 - x^2}$. Set $x = a \sin t$.

$$I = a^4 \int \sin^2 t \cos^2 t\,dt = \frac{a^4}{4}\int \sin^2 2t\,dt$$

$$= \frac{a^4}{8}\int (1 - \cos 4t)dt = \frac{a^4}{8}\left(t - \frac{1}{4}\sin 4t\right) + C$$

$$= \frac{a^4}{8}\arcsin\frac{x}{a} + \left(\frac{1}{4}x3 - \frac{a^2}{8}x\right)\sqrt{a^2 - x^2} + C.$$

(d) $\dfrac{1}{x^2\sqrt{x^2 - a^2}}$. Set $x = a \cosh u$.

$$I = \frac{1}{a^2}\int \frac{du}{\cosh^2 u} = \frac{1}{a^2}\tanh u + C$$

$$= \frac{\sqrt{a^2 - x^2}}{a^2 x} + C.$$

(e) $\dfrac{x}{(x^2 + a^2)\sqrt{x^2 - b^2}}$. Set $x^2 - b^2 = t^2$.

$$I = \int \frac{dt}{a^2 + b^2 + t^2} = \frac{1}{\sqrt{a^2 + b^2}}\arctan\frac{t}{\sqrt{a^2 + b^2}} + C$$

$$= \frac{1}{\sqrt{a^2 + b^2}}\arctan\sqrt{\frac{x^2 - b^2}{a^2 + b^2}} + C.$$

(f) $\dfrac{1}{(x^2 + a^2)\sqrt{a^2x^2 + 1}}$. Set $x = \frac{1}{a}\sinh t$.

$$I = \int \frac{a\,dt}{a^4 + \sinh^2 t} = a \int \frac{1}{a^4 + (1 - a^4)\tanh^2 t} \; \frac{1}{\cosh^2 t}\,dt$$

$$= a \int \frac{ds}{a^4 + (1 - a^4)s^2}$$

where $s = \tanh t$.

If $a = 1$, then $I = s + C = \dfrac{x}{\sqrt{x^2 + 1}} + C$.

If $a = 0$, then $I = -\dfrac{1}{x} + C$.

If $0 < a < 1$, then

$$I = \frac{1}{a\sqrt{1 - a^4}} \arctan \frac{\sqrt{1 - a^4}}{a^2}\, s + C$$

$$= \frac{1}{a\sqrt{1 - a^4}} \arctan \frac{x}{a} \sqrt{\frac{1 - a^4}{1 + a^2x^2}} + C.$$

If $a > 1$, then for $b = \dfrac{a^2}{\sqrt{a^4 - 1}}$,

$$I = \frac{b^2}{a^3} \int \frac{ds}{b^2 - s^2} = \frac{b}{2a^3} \log \left|\frac{b + s}{b - s}\right| + C$$

$$= \frac{b}{2a^3} \log \left|\frac{b\sqrt{1 + a^2x^2} + ax}{b\sqrt{1 + a^2x^2} - ax}\right|.$$

(g) $\dfrac{x + 2}{\sqrt{m^2 \pm x^2}} = \dfrac{x}{\sqrt{m^2 \pm x^2}} + \dfrac{2}{\sqrt{m^2 \pm x^2}}$.

Plus sign: $I = \sqrt{m^2 + x^2} + 2 \operatorname{argsinh} x + C$.

Minus sign: $I = -\sqrt{m^2 - x^2} + 2 \arcsin x + C$.

(h) $x^3 \sqrt{(4 - x^2)^5}$. Set $\sqrt{4 - x^2} = t$.

$$I = \int (t^2 - 4)t^6\,dt = \frac{-(4 - x^2)^{7/2}(8 + 7x^2)}{63} + C.$$

(i) $\frac{1}{\sqrt{a^2x - x^2}} = \frac{1}{\sqrt{\frac{a^2}{4} - (x - \frac{a^2}{2})^2}}$.

Set $x = \frac{a^2}{2}(\sin\theta + 1)$.

$$I = \arcsin\left(\frac{2x}{a^2} - 1\right) + C$$

(Alternatively, set $x = a^2 \sin^2\psi$ to obtain $I = 2\psi = 2 \arcsin \frac{\sqrt{x}}{a}$. This is suggested by the preliminary substitution $x = t^2$.)

(j) $\frac{x^2 + ax + b}{x^2 + 1} = \frac{(x^2 + 1) + ax + (b - 1)}{x^2 + 1}$.

$$I = x + \frac{a}{2}\log(x^2 + 1) + (b - 1)\arctan x + C.$$

(k) $\sqrt{a^2x + x^2} = \sqrt{(x + \frac{a^2}{2})^2 - \frac{a^4}{4}}$.

Set $x + \frac{a^2}{2} = \frac{a^2}{2}\cosh z$..

$$I = \frac{a^4}{4}\int \sinh^2 z\, dz = \frac{a^4}{8}\int(\cosh 2z - 1)dz$$

$$= (x + \frac{a^2}{2})\sqrt{a^2x + x^2} - \frac{a^4}{8}\operatorname{argcosh}\left(\frac{2x}{a^2} + 1\right) + C.$$

2. Let $R(x,y)$ denote a rational function in x and y. Reduce the following integrals to integrals of rational functions.

(a) $\int R(x, \sqrt{ax + b})dx$,, $a \neq 0$. Set $\sqrt{ax + b} = t$.

$$I = \frac{2}{a}\int R\left(\frac{t^2 - b}{a}, t\right)t\, dt.$$

(b) $\int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right)dx$, n - an integer, $ad - bc \neq 0$.

Set $\sqrt[n]{\frac{ax + b}{cx + d}} = t$.

$$I = \int n(ad - bc)\frac{t^{n-1}}{(a - ct^n)^2} R\left\{\frac{dt^n - b}{a - ct^n}, t\right\} dt.$$

3. Using the result of Number 2, integrate $\dfrac{x}{\sqrt{ax + b} + (\sqrt{ax + b})^3}$.

$$I = \frac{2}{a^2}\int \frac{(t^2 - b)t}{t + t^3}\,dt = \frac{2}{a^2}\int \left(1 - \frac{1 + b}{1 + t^2}\right)dt$$

$$= \frac{2}{a^2}\left[\sqrt{ax + b} - (1 + b)\arctan\sqrt{ax + b}\right] + C.$$

4. Reduce to rational form, $\displaystyle\int \frac{dx}{\sqrt{\frac{1 - x}{1 + x}} + \sqrt[4]{\frac{1 - x}{1 + x}}}$.

Use the method of Number 2(b).

$$I = -8\int \frac{t^2\,dt}{(1 + t)(1 + t^4)^2}.$$

5. Express as elementary functions,

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$.

First note that $\dfrac{1}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}} = \dfrac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{2}$.

$$I = \frac{x}{4}\left(\sqrt{x^2 + 1} - \sqrt{x^2 - 1}\right) + \frac{1}{4}\log\left|(x + \sqrt{x^2 + 1})(x + \sqrt{x^2 - 1})\right| + C.$$

(b) $\displaystyle\int \frac{dx}{1 + \sin x} = \int \frac{1 - \sin x}{\cos^2 x}\,dx = \tan x - \frac{1}{\cos x} + C.$

(c) $\displaystyle\int \frac{dx}{1 - \cos 2x} = \int \frac{1 + \cos 2x}{\sin^2 2x}\,dx = -\frac{1}{2}\left(\cot 2x + \frac{1}{\sin 2x}\right) + C.$

(d) $\displaystyle\int \frac{dx}{x\sqrt[4]{1 + x^4}}$. Set $t = \sqrt[4]{1 + x^4}$; then

$$I = \int \frac{t^2}{t^4 - 1}\,dt = \frac{1}{2}\int \frac{(t^2 + 1) + (t^2 - 1)}{t^4 - 1}\,dt$$

$$= \frac{1}{2}\int \left[\frac{1}{t^2 - 1} + \frac{1}{t^2 + 1}\right]dt$$

$$= \frac{1}{4}\log\left|\frac{t - 1}{t + 1}\right| + \frac{1}{2}\arctan t + C$$

$$= \frac{1}{4}\log\frac{\sqrt[4]{1 + x^4} - 1}{\sqrt[4]{1 + x^4} + 1} + \frac{1}{2}\arctan\sqrt[4]{1 + x^4} + C.$$

(e) $\displaystyle\int \frac{dx}{\sqrt[4]{1 + x^4}}$. Set $x = \frac{1}{u}$; then

$$I = -\int \frac{du}{u\sqrt[4]{1 + u^4}}$$

and the problem is reduced to (d).

6. (a) The integral $\displaystyle\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\,dx$ where $P(x)$ is a polynomial of degree n and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can also be reduced to the integration of $\dfrac{1}{\sqrt{ax^2 + 2bx + c}}$; namely, for some polynomial Q of degree $n - 1$ and constant k,

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D_x\{Q(x)\sqrt{ax^2 + 2bx + c}\} + \frac{k}{\sqrt{ax^2 + 2bx + c}}.$$

Show how to find Q and k.

Since

$$D_x(Q(x)\sqrt{ax^2 + 2bx + c}) = \frac{Q'(x)(ax^2 + 2bx + c) + Q(x)(ax + b)}{\sqrt{ax^2 + 2bx + c}},$$

the polynomial Q and the constant k must sastisfy

$$(1) \quad P(x) \equiv Q(x)(ax + b) + Q'(x)(ax^2 + 2bx + c) + k.$$

The constant k and the coefficients of $Q(x)$ can be found from the coefficients of $P(x)$ as follows. Set $P(x) = \sum_{\nu=0}^{n} p_\nu x^{n-\nu}$ and $Q(x) = \sum_{\nu=0}^{n-1} q_\nu x^{n-1-\nu}$. Equate coefficients of like powers on the right and left in (1) beginning with the coefficients of x^n:

$$p_0 = aq_0 + a(n - 1)q_0$$

to obtain

$$q_0 = \frac{p_0}{an} .$$

Thus the leading coefficient of $Q(x)$ is determined. Next, $p_1 = (aq_1 + bq_0) + [a(n - 2)q_1 + 2b(n - 1)q_0]$, or $a(n - 1)q_1 = p_1 - b(2n - 1)q_0$. Similarly all succeeding coefficients are determined step-by-step from the preceding ones:

$$a(n - \nu)q_\nu = p_\nu - b(2n - 2\nu - 1)q_{\nu-1} - c(n - \nu + 1)q_{\nu-2},$$

for $\nu = 2, \ldots, n - 1$. Finally, for the constant k,

$$k = p_0 - bq_0 - cq_1 .$$

(b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1 - t^2}}$.

Set $Q(t) = q_0 t^4 + q_1 t^3 + q_2 t^2 + q_3 t + q_4$ and $P(t) = t^5 - t^3 + t$.

$$D(Q(t)\sqrt{1 - t^2}) = \frac{1}{\sqrt{1 - t^2}} [-5q_0 t^5 - 4q_1 t^4 + (4q_0 - 3q_2)t^3 + (3q_1 - 2q_3)t^2 + (2q_2 - q_4)t + q_3].$$

Now solve for the coefficients in succession to obtain

$$q_0 = -\frac{1}{5}, \quad q_1 = 0, \quad q_2 = \frac{1}{15}, \quad q_3 = 0,$$
$$q_4 = -\frac{13}{15}, \quad k = 0.$$

Consequently,

$$I = \int \frac{t^5 - t^3 + t}{\sqrt{1 - t^2}}\, dt = -\frac{1}{15}(3t^4 - t^2 + 13)\sqrt{1 - t^2} + C.$$

(c) Find the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

Set $t = \sin\theta$. Then

$$I = \int (\sin^4\theta - \sin^2\theta + 1)\sin\theta\, d\theta$$

$$= \int [(1 - \cos^2\theta)^2 - (1 - \cos^2\theta) + 1]\sin\theta\, d\theta$$

$$= \int [(\cos^4\theta - \cos^2\theta + 1]\sin\theta\, d\theta$$

$$= -\frac{1}{5}\cos^5\theta + \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$$= -\frac{1}{15}(3t^4 - t^3 + 13)\sqrt{1 - t^2} + C.$$

which is the result obtained in (b). If even powers appeared in $P(t)$ the work in (c) would be more complicated while the work in (b) would not change greatly. Furthermore the method of (a) eliminates the repetitive use of the binomial theorem when P is of high degree.

7. Integrate

(a) $\dfrac{1}{\sin x}$

Use $x = 2\arctan t$ to obtain

$$I = \int \frac{dt}{t} = \log\left|\tan\frac{x}{2}\right| + C.$$

(b) $\dfrac{1}{\cos x}$ (by a method other than that of Example A4-1d).

Use $x = 2\arctan t$ to obtain

$$I = \int \frac{dt}{1 - t^2} = \frac{1}{2}\int \left[\frac{1}{1 - t} + \frac{1}{1 + t}\right]dt$$

$$= \frac{1}{2}\log\left|\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right| + C.$$

Alternatively, from $\cos x = \sin u$, where $u = x + \frac{\pi}{2}$, obtain from the solution of Part (a),

$$I = \int \frac{du}{\sin u} = \log\left|\tan \frac{1}{2}\left(x + \frac{\pi}{2}\right)\right| + C.$$

A4-2. Integration by Parts

THEOREM A4-2. When we say "ϕ and ψ are inverses" we mean ϕ is the inverse of ψ and ψ is the inverse of ϕ. (See, also, Exercises A4-2, No. 2.)

Example A4-2b. It might be appropriate to first consider $\int \arcsin x \, dx$. We observe that

$$\arcsin x = (\arcsin x)(1) = (\arcsin x)\frac{dx}{dx}.$$

Thus $\arcsin x = u \frac{dv}{dx}$ where $u = \arcsin x$ and $v = x$. Integrating by parts we get

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

$$= x \arcsin x + \sqrt{1 - x^2} + C.$$

It should be noted that the discussion in the text establishes the integrability of $x^{-1} \arcsin x$.

Solutions Exercises A4-2

1. Integrate the following functions.

In these solutions the integral of the problem is written $I = \int u \, dv$.

(a) $x \sin 3x$ $\qquad I = -x \frac{\cos 3x}{3} + \frac{\sin 3x}{9} + C.$

(b) $x \cdot 5^x$. $\qquad I = \frac{x \cdot 5^x}{\log 5} - \frac{5^x}{(\log 5)^2} + C.$

(c) $x^3 e^{-2x}$ $\qquad$ Set $dv = e^{-2x}dx$ and integrate by parts three times.

$$I = -e^{-2x}\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) + C.$$

(d) $\sqrt{x} \log ax$ $\qquad$ Set $u = \log ax$, $dv = \sqrt{x}\, dx$.

$$I = \frac{2}{3} x^{3/2} \left(\log ax - \frac{2}{3}\right) + C.$$

Alternatively, set $t = ax$ and use the result of Example A4-2c.

(e) $\log^2 bx$

Set $u = \log^2 bx$.

$$I = x \log^2 bx - 2\int \log bx \, dx + C$$
$$= x \log^2 bx - 2x(\log bx - 1) + C$$

where the integration of Example 10-4a is used at the end.

(f) $\log^3 x$

Set $u = \log^3 x$. Apply (e).

$$I = x(\log^3 x - 3 \log^2 x + 6 \log x - 6) + C.$$

(g) $\arccos 7x$

Set $u = \arccos 7x$. Then

$$I = x \arccos 7x - \frac{\sqrt{1 - 49x^2}}{7} + C.$$

(h) $\operatorname{argsinh} ax$

Set $u = \operatorname{argsinh} ax$.

$$I = x \operatorname{argsinh} ax - \frac{1}{a}\sqrt{1 + a^2x^2} + C.$$

(i) $\operatorname{argtanh} bx$

Set $u = \operatorname{argtanh} bx$.

$$I = x \operatorname{argtanh} bx + \frac{1}{b} \log(1 - b^2x^2) + C.$$

(j) $\operatorname{argtanh} \sqrt{bx}$

Set $u = \operatorname{argtanh} \sqrt{bx}$.

$$I = x \operatorname{argtanh} \sqrt{bx} - \frac{1}{2}\int \frac{\sqrt{bx}}{1 - bx} \, dx$$
$$= x \operatorname{argtanh} \sqrt{bx} - \frac{1}{b}\int \frac{t^2}{1 - t^2} \, dt$$

where $t = \sqrt{bx}$. From $\frac{t^2}{1 - t^2} = \frac{1}{1 - t^2} - 1$, obtain

$$I = \left(x - \frac{1}{b}\right) \operatorname{argtanh} \sqrt{bx} + \sqrt{\frac{x}{b}} + C.$$

(k) $\arctan \sqrt[3]{x}$ Set $u = \arctan \sqrt[3]{x}$.

$$I = x \arctan \sqrt[3]{x} - \frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx .$$

Set $x = z^3$, then

$$\begin{aligned} \frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx &= \int \frac{z^3}{1 + z^2}\, dz \\ &= \frac{1}{2}\int \frac{t}{1 + t}\, dt \qquad (\text{where } t = z^2) \\ &= \frac{1}{2}\int \left(1 - \frac{1}{1 + t}\right) dt \\ &= \frac{1}{2}(t - \log|1 + t|) . \end{aligned}$$

Consequently,

$$I = x \arctan \sqrt[3]{x} - \frac{x^{2/3}}{2} - \log(x^{2/3} + 1) + C.$$

(ℓ) $x \arctan x$ Set $u = \arctan x$, $dv = x\, dx$.

$$I = \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{x^2}{1 + x^2}\, dx .$$

From $\dfrac{x^2}{1 + x^2} = 1 - \dfrac{1}{1 + x^2}$, obtain

$$I = \frac{1}{2}\{(x^2 + 1)\arctan x - x\} + C .$$

(m) $\dfrac{\arccos \frac{x}{m}}{\sqrt{x + m}}$ Set $u = \arccos \dfrac{x}{m}$, $v = 2\sqrt{x + m}$.

$$\begin{aligned} I &= 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{\sqrt{x + m}}{\sqrt{m^2 - x^2}}\, dx. \\ &= 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{dx}{\sqrt{m - x}} \\ &= 2\sqrt{x + m} \arccos \frac{x}{m} - 4\sqrt{m - x} + C . \end{aligned}$$

(n) $x \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$I = \frac{x^2}{4} - \frac{1}{2}\int x \cos 2x \, dx.$$

Set $u = x$, $dv = \cos 2x \, dx$, then

$$I = \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.$$

(o) $x^2 \sin x$ Set $u = x^2$, $v = -\cos x$.

$$I = -x^2 \cos x + \int 2x \cos x \, dx.$$

Now set $u = x$, $v = \sin x$.

$$\int 2x \cos x \, dx = 2x \sin x - 2\int \sin x \, dx;$$

whence,

$$I = -(x^2 + 2)\cos x + 2x \sin x + C.$$

(p) $x^2 \arcsin ax$ Set $u = \arcsin ax$, $v = \frac{x^3}{3}$.

$$I = \frac{x^3}{3} \arcsin ax - \frac{a}{3}\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx.$$

Now set $z = \sqrt{1 - a^2x^2}$, $x^2 = \frac{1 - z^2}{a^2}$, $x\,dx = -\frac{z\,dz}{a^2}$. Then

$$\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx = -\frac{1}{a^4}\int (1 - z^2)\,dz;$$

whence

$$I = \frac{x^3}{3} \arcsin ax - \frac{\sqrt{1 - a^2x^2}}{3a^3} + \frac{(\sqrt{1 - a^2x^2})^3}{9a^3} + C.$$

(q) $\cos^3 2x$ Follow the method of Example A4-2g.

More simply, note that

$$\cos^3 2x = (1 - \sin^2 2x)\cos 2x$$

$$I = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C.$$

(r) $\sin^5 x$ Either follow the method of Example A4-2g or use

$$\sin^5 x = (1 - \cos^2 x)^2 \sin x.$$

$$I = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C.$$

(s) $\sin(\log ax)$ Set $u = \sin(\log ax)$.

$$I = x \sin(\log ax) - \int a \cos(\log ax)dx.$$

Similarly,

$$\int \cos(\log ax)dx = x \cos(\log ax) + aI;$$

whence,

$$I = x \sin(\log ax) - ax \cos(\log ax) - a^2 I$$

and

$$I = \frac{x}{1 + a^2} \{\sin(\log ax) - a \cos(\log ax)\} + C.$$

(t) $x \tan^2 x$ Set $u = x$, $v = \tan x - x$.

$$I = x \tan x - x^2 - \int (\tan x - x)dx$$

$$= x \tan x + \log |\cos x| - \frac{x^2}{2} + C.$$

(u) $(\arcsin x)^2$. Substitute $x = \sin t$ to obtain $I = \int t^2 \cos t\, dt$ and integrate by parts twice. Alternatively, set $u = (\arcsin x)^2$ to obtain

$$I = x(\arcsin x)^2 - 2\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx.$$

Repeat, to obtain

$$\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx = -\sqrt{1 - x^2} \arcsin x + \int dx.$$

$$I = x(\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C.$$

(v) $\sin ax \cos bx$, $(a^2 \neq b^2)$. Set $u = \sin ax$, $v = \dfrac{\sin bx}{b}$.

$$I = \frac{\sin ax \sin bx}{b} - \frac{a}{b}\int \cos ax \sin bx\, dx.$$

Now set $u = \cos ax$, $v = -\dfrac{\cos bx}{b}$.

$$\int \cos ax \sin bx\, dx = -\frac{\cos ax \cos bx}{b} - \frac{a}{b} I.$$

Consequently,

$$I = \frac{\sin ax \sin bx}{b} + \frac{a}{b^2} \cos ax \cos bx + \frac{a^2}{b^2} I;$$

whence

$$I = \frac{1}{b^2 - a^2} [a \cos ax \cos bx + b \sin ax \sin bx].$$

More simply, note that

$$\sin ax \cos bx = \frac{1}{2}[\sin(a + b)x + \sin(a - b)x].$$

Hence

$$I = -\frac{1}{2}\left[\frac{\cos(a + b)x}{a + b} + \frac{\cos(a - b)x}{a - b}\right] + C.$$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where f and g have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverse functions.

Let F be the inverse of f, G the inverse of g. The functions

$$\phi = fG : v \to fG(v) = f(x) = u$$
$$\psi = gF : u \to gF(u) = g(x) = v$$

are inverses.

3. Verify as alleged after Example A4-2b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if f is any inverse circular or hyperbolic function, or if f is the logarithmic function.

Set $u = f(x)$, $v = \frac{x^{n+1}}{n+1}$. Let F be the inverse of F. Then

$$v = \frac{F(u)^{n+1}}{n+1}.$$

If F is a trigonometric or hyperbolic function, then $\int v\,du$ is reducible to the integral of a rational function by Theorem A4-1b or Exercises A4-1, Number 10.

If $u = \log x$, then $F(u) = e^u$. Here,

$$\int v\,du = \int \frac{e^{(n+1)u}}{n+1}\,du = \frac{e^{(n+1)u}}{(n+1)^2} + C$$
$$= \frac{x^{n+1}}{(n+1)^2} + C.$$

Thus, explicitly,

$$\int x^n \log x\,dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C.$$

4. Establish recurrence relations for each of the following (in each case m and n are positive integers).

(a) $\int \sin^n x \, dx$

Proceed as in Example A4-2g. Otherwise use the result of the example:

$$I_n = \int \cos^n(x - \tfrac{\pi}{2})dx$$
$$= -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}.$$

(b) $\int x^m \log^n x \, dx$ Set $u = \log^n x$, $v = \dfrac{x^{m+1}}{m+1}$.

$$I_{n,m} = \frac{x^{m+1}}{m+1} \log^n x - \frac{n}{m+1} I_{n-1,m}.$$

$$I_{0,m} = \frac{x^{m+1}}{m+1}.$$

(c) $\int \sin^m x \cos^n x \, dx$ Set $u = \cos^{n-1} x$, $v = \dfrac{\sin^{m+1} x}{m+1}$.

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2}x \cos^{n-2}x \, dx..$$

But, since $\sin^{m+2}x = (1 - \cos^2 x)\sin^m x$,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \{I_{m,n-2} - I_{m,n}\};$$

whence,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}.$$

Note that $I_{m,0}$ is given by (a).

It is also possible to reduce first m then n. Instead of proceeding by the given method, use

$$\sin^m x \cos^n x = (-1)^m \cos^m (x - \tfrac{\pi}{2}) \sin^m (x - \tfrac{\pi}{2})$$

to obtain by the preceding result

$$I_{n,m} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}.$$

(d). $\int x^n \arctan x \, dx$ Set $u = \arctan x$, $v = \frac{x^{n+1}}{n + 1}$. Then

$$(1) \qquad I_n = \frac{x^{n+1}}{n + 1} \arctan x - \frac{1}{n + 1} \int \frac{x^{n+1}}{1 + x^2} dx.$$

But

$$(2) \qquad \frac{x^{n+1}}{1 + x^2} = \frac{(1 + x^2)x^{n-1} - x^{n-1}}{1 + x^2}.$$

Insert this in (1) to obtain

$$I_n = \frac{x^{n+1}}{n + 1} \arctan x - \frac{x^n}{n(n + 1)} + \frac{1}{n + 1} \int \frac{x^{n-1}}{1 + x^2} dx$$

where the integral is given by (1) in terms of I_{n-2}. From this,

$$I_n = \frac{x^{n-1}}{n + 1} \{(1 + x^2) \arctan x - \frac{x}{n}\} - \frac{n - 1}{n + 1} I_{n-2}.$$

Alternatively, reduce the integral in (1) by (2)

$$J_{n+1} = \int \frac{x^{n+1}}{1 + x^2} dx = \frac{x^n}{n} - J_{n-1}$$

$$= \frac{x^n}{n} - \frac{x^{n-2}}{n - 2} + \frac{x^{n-4}}{n - 4} - \dots .$$

If n is odd then the sum terminates with $(-1)^{\frac{n-1}{2}} J_1$ where $J_1 = \int \frac{x}{1 + x^2} dx = \frac{1}{2} \log(1 + x^2)$. If n is even then the sum terminates with $(-1)^{n/2} J_0$ where $J_0 = \int \frac{dx}{1 + x^2} = \arctan$

(e) $\int x^n \operatorname{argsinh} x \, dx$ First substitute $x = \sinh t$ to obtain

$$I_n = \int t \sinh^n t \cosh t \, dt.$$

Now set $u = t$, $v = \frac{\sinh^{n+1} t}{n + 1}$; then

$$(1) \qquad I_n = t \frac{\sinh^{n+1} t}{n + 1} - \frac{1}{n + 1} \int \sinh^{n+1} t \, dt.$$

Then proceed as in Example A4-2g to obtain

$$J_{n+1} = \int \frac{\sinh^{n+1} t}{n+1}\, dt$$

$$= \frac{\sinh^n t \cosh t}{(n+1)^2} - \frac{n(n-1)}{(n+1)^2} J_{n-1}.$$

But, from (1)

$$J_{n-1} = \frac{t \sinh^{n-1} t}{n-1} - I_{n-2}.$$

Combine these results to obtain

$$I_n = \frac{t \sinh^{n+1} t}{n+1} - \frac{\sinh^n t \cosh t}{(n+1)^2} + \frac{n}{(r+1)^2} t \sinh^{n-1} t - \frac{n}{(n+1)^2} I_{n-2}$$

$$= \frac{x^{n-1}}{n+1}\left(x^2 + \frac{n}{n+1}\right) \operatorname{argsinh} x - \frac{x^n\sqrt{1+x^2}}{(n+1)^2} - \frac{n}{(n-1)^2} I_{n-2}.$$

(f) $\int x^n \operatorname{argtanh} x\, dx$ Proceed as in (d).

$$I = \frac{x^{n+1}}{n+1} \operatorname{argtanh} x - \frac{1}{n+1} J_{n+1},$$

where

$$J_{n+1} = \int \frac{x^{n+1}}{1-x^2}\, dx = \int \frac{x^{n-1} - (1-x^2)x^{n-1}}{1-x^2}$$

$$= J_{n-1} - \frac{x^n}{n}.$$

(g) $\int x^n e^{ax}\, dx$ Set $u = x^n$, $v = \frac{e^{ax}}{a}$.

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n-1}{a} I_{n-1}.$$

(h) $\int x^n \arcsin x\, dx$ Proceed as in (e).

$$I_n = \frac{x^{n-1}}{n+1}\left(x^2 - \frac{n}{n+1}\right)\arcsin x + \frac{x^n\sqrt{1-x^2}}{(n+1)^2} + \frac{n(n-1)}{(n+1)^2} I_{n-2}.$$

(i) $\int \frac{1}{\sin^n x}\,dx$ $\qquad u = \frac{1}{\sin^{n-2} x}, \quad v = -\frac{\cos x}{\sin x}.$

Then
$$I_n = \frac{\cos x}{\sin^{n-1} x} - (n-2)\int \frac{\cos^2 x}{\sin^n x}\,dx.$$

Set $\cos^2 x = 1 - \sin^2 x$ in the last integral to obtain
$$I_n = -\frac{1}{n-1}\,\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\,I_{n-2}.$$

(j) $\int \frac{e^x}{x^n}\,dx$ $\qquad$ Set $u = e^x$, $dv = \frac{1}{x^n}\,dx$.
$$I_n = -\frac{1}{n-1}\,\frac{e^x}{x^n} + \frac{1}{n-1}\,I_{n-1}.$$

(k) $\int x^n \cos x\,dx$ $\qquad$ Set $u = x^n$, $v = \sin x$.
$$I_n = x^n \sin x - n\int x^{n-1} \sin x\,dx.$$

Now set $u = x^{n-1}$, $v = -\cos x$.
$$\int x^{n-1} \sin x\,dx = -x^{n-1} \cos x + (n-1)\int x^{n-2} \cos x\,dx.$$

Consequently,
$$I_n = x^n \sin x + n\,x^{n-1} \cos x - n(n-1)I_{n-2}.$$

For n even, the expansion of I_n terminates with $I_0 = \sin x + C$; for n odd, the expansion ends with $I_1 = x \sin x + \cos x + C$.

A4-3. Integration of Rational Functions

The integral of a rational function is an elementary function, but the computation of such an integral may involve considerable labor depending on the complexity of the function at hand.

Every polynomial with real coefficients has a unique factorization of the form given by Equation (4), but to obtain this form one must first find the zeros of Q. The polynomials which appear in exercises in various textbooks are manufactured artifically for purposes of illustration: they are either given in a factored form, or have zeros which can be found easily. In problems arising in applications this is often not the case.

The method of equated coefficients (Example A4-3c) is, of course, applicable where the roots of Q are all real of multiplicity 1 (as in (7)).

In Example A4-3d since the integrand may be decomposed into the sum of rational functions $\frac{a}{x}$, $\frac{b}{x^2}$, $\frac{cx}{x^2 + 4}$, $\frac{d}{(\frac{x}{2})^2 + 1}$ we know that the integral must be of the form stated.

Example TCA4-3a. Consider $I = \int \frac{dx}{x^3(x - 1)}$. Note that

$$\frac{1}{x^3(x - 1)} = \frac{(1 - x)(1 + x + x^2) + x^3}{x^3(x - 1)} = \frac{1}{x - 1} - \frac{1}{x^3} - \frac{1}{x^2} - \frac{1}{x}.$$

Thus we composed the integrand into the sum of simpler rational functions which may be integrated at sight:

$$I = \log |x - 1| - \log |x| + \frac{1}{x} + \frac{1}{2x^2} + C.$$

Example TCA4-3b. The integral $I = \int \frac{2x}{x^2 + x + 1} dx$ is computed by decomposing the integrand into a sum of derivatives of two known functions. We have

$$\frac{2x}{x^2 + x + 1} = \frac{2x + 1}{x^2 + x + 1} - \frac{1}{x^2 + x + 1}$$

$$= D \log (x^2 + x + 1) - D(\frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}}).$$

Thus $I = \log (x^2 + x + 1) - \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C.$

Solutions Exercises A4-3

1. Integrate the following.

Parts (a) to (k) are simple and an ad hoc approach is probably simpler and quicker than a straight forward application of theory.

(a) $\dfrac{x + 2}{x^2 + 3x + 1}$

Set $x^2 + 3x + 1 = (x - a)(x - b)$ where

$$a = \frac{-3 + \sqrt{5}}{2} \quad \text{and} \quad b = \frac{-3 - \sqrt{5}}{2}.$$

$$\frac{x + 2}{(x - a)(x - b)} = \frac{(x - a) + 2 + a}{(x - a)(x - b)} = \frac{1}{x - b} + \frac{2 + a}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right)$$

$$= \frac{2 + b}{b - a}\,\frac{1}{x - b} - \frac{2 + a}{b - a}\,\frac{1}{x - a},$$

where

$$\frac{2 + b}{b - a} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right) \qquad \frac{2 + a}{b - a} = -\frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right).$$

$$I = \frac{1}{2}\left(1 - \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 + \sqrt{5}}{2}\right| + \frac{1}{2}\left(1 + \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 - \sqrt{5}}{2}\right| + C.$$

(b) $\dfrac{x^3}{(x^2 + 3x - 10)}$

By long division,

$$\frac{x^3}{x^2 + 3x - 10} = x - 3 + \frac{19x - 30}{x^2 + 3x - 10}.$$

But

$$\frac{19x - 30}{x^2 + 3x - 10} = \frac{19x - 30}{(x + 5)(x - 2)}$$

$$= \frac{8}{7(x - 2)} + \frac{125}{7(x + 5)}.$$

Consequently,

$$I = \frac{x^2}{2} - 3x + \frac{8}{7}\log|x - 2| + \frac{125}{7}\log|x + 5| + C.$$

(c) $\dfrac{x^3}{x^2 + 2ax + b^2}$, $(b > |a|)$

$$\frac{x^3}{x^2 + 2ax + b^2} = x - 2a + \frac{(2a^2 - \frac{b^2}{2})[2(x + a)] + (3ab^2 - 4a^3)}{x^2 + 2ax + b^2}$$

$$I = \frac{x^2}{2} - 2ax + (2a^2 - \frac{b^2}{2})\log(x^2 + 2ax + b^2) + \frac{3ab^2 - 4a^3}{\sqrt{b^2 - a^2}} \arctan \frac{x + a}{\sqrt{b^2 - a^2}} + C.$$

(d) $\dfrac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$ (consider the cases $a \neq b$ and $a = b$).

If $a \neq b$,

$$I = x + \frac{a^2 + \alpha a + \beta}{a - b} \log |x - a| + \frac{b^2 + \alpha b + \beta}{b - a} \log |x - b| + C.$$

If $a = b$,

$$I = x + (\alpha + 2a)\log(x - a) - \frac{a^2 + \alpha a + \beta}{x - a} + C.$$

(e) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$, ($a$, b, c distinct).

$$I = \frac{a^2}{(a - b)(a - c)} \log |x - a| + \frac{b^2}{(b - a)(b - c)} \log |x - b|$$
$$+ \frac{c^2}{(c - a)(c - b)} \log |x - c| + C.$$

(f) $\dfrac{x^3 + 1}{x^3 - 1} = 1 + \dfrac{2}{(x - 1)(x^2 + x + 1)} = 1 + \dfrac{2}{3(x - 1)} - \dfrac{2x + 4}{3(x^2 + x + 1)}$

$$I = x + \frac{2}{3}\log|x - 1| - \frac{1}{3}\log(x^2 + x + 1) - \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C$$

(g) $\dfrac{1}{x^3 + a^3} = \dfrac{1}{(x + a)(x^2 - ax + a^2)} = \dfrac{1}{3a^2(x + a)} - \dfrac{x - 2a}{3a^2(x^2 - ax + a^2)}$

$$I = \frac{1}{3a^2} \log|x + a| - \frac{1}{6a^2} \log(x^2 - ax + a^2) - \frac{5\sqrt{3}}{9a^2} \arctan \frac{2x - a}{a\sqrt{3}} + C$$

(h) $$\frac{(x + 2)^2}{x(x - 1)^2} = \frac{9}{(x - 1)^2} - \frac{3}{x - 1} + \frac{4}{x}$$

$$I = -\frac{9}{x - 1} - 3 \log |x - 1| + 4 \log |x| + C$$

(i) $$\frac{1}{x^4 - 1} = \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} - \frac{1}{2(x^2 + 1)} .$$

$$I = \frac{1}{4} \log \left|\frac{x - 1}{x + 1}\right| - \frac{1}{2} \arctan x + C$$

(j) $$\frac{x^2}{x^4 - 1} = \frac{x^2 - 1}{x^4 + 1} + \frac{1}{x^4 - 1} = \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x^2 + 1)}$$

(see (i)).

$$I = \frac{1}{4} \log \left|\frac{x - 1}{x + 1}\right| + \frac{1}{2} \arctan x + C$$

(k) $$\frac{1}{x^4 + x^6} = \frac{1}{x^4(1 + x^2)} = \frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1 + x^2} .$$

$$I = -\frac{1}{3x^3} + \frac{1}{x} + \arctan x + C$$

(ℓ) $$\frac{x^4}{x^4 + 1} = 1 - \frac{1}{x^4 + 1}$$

Since $x^4 + 1 > 0$ for all x, $x^4 + 1$ cannot have linear factors. Set

$$(x^4 + 1) = (x^2 + ax + b)(x^2 + cx + d).$$

Equate coefficients of x^3 to obtain $c = -a$; of x,, to obtain $b = d$ ($a = 0$ is not possible), of x^0 to obtain $b = \pm 1$, of x^2, to obtain $a^2 = \pm 2$, hence only $b = 1$ is possible and $a = \sqrt{2}$.

$$(x^4 + 1) = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$$

Set

$$\frac{1}{x^4 + 1} = \frac{Ax + B}{x^2 + x\sqrt{2} + 1} + \frac{Cx + D}{x^2 - x\sqrt{2} + 1} .$$

Use the method of undetermined coefficients to obtain $A = -C = \frac{1}{2\sqrt{2}}$, $B = D = \frac{1}{2}$; whence,

$$\frac{1}{x^4 + 1} = \frac{1}{4\sqrt{2}} \frac{2x + \sqrt{2}}{x^2 + x\sqrt{2} + 1} - \frac{1}{4\sqrt{2}} \frac{2x - \sqrt{2}}{x^2 - x\sqrt{2} + 1}$$

$$+ \frac{1}{4} \frac{1}{x^2 + x\sqrt{2} + 1} - \frac{1}{4} \frac{1}{x^2 - x\sqrt{2} + 1} .$$

Consequently,

$$I = x - \frac{1}{4\sqrt{2}} \log \left| \frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} \right|$$

$$- \frac{1}{2\sqrt{2}} \{\arctan(1 + x\sqrt{2}) + \arctan(1 - x\sqrt{2})\} + C.$$

(m) $\dfrac{1}{x^6 - 1}$

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$$

$$\frac{1}{x^6 - 1} = \frac{1}{6}\left\{\frac{1}{x - 1} - \frac{1}{x + 1} - \frac{x + 2}{x^2 + x + 1} + \frac{x - 2}{x^2 - x + 1}\right\}$$

$$I = \frac{1}{6} \log \left|\frac{x - 1}{x + 1}\right| + \frac{1}{12} \log \frac{x^2 - x + 1}{x^2 + x + 1} - \frac{2}{\sqrt{3}} \arctan \frac{2}{\sqrt{3}}(x + \tfrac{1}{2})$$

$$- \frac{2}{\sqrt{3}} \arctan \frac{2}{\sqrt{3}}(x - \tfrac{1}{2}) + C$$

As a special challenge you may wish to ask for $I = \int \frac{dx}{1 + x^6}$ which Leibniz failed to represent in elementary terms. For this, note that

$$1 + x^6 = (1 + x^2)(1 - x^2 + x^4)$$

$$= (1 + x^2)(1 - x\sqrt{3} + x^2)(1 + x\sqrt{3} + x^2).$$

In this case,

$$I = \frac{\arctan x}{3} + \frac{1}{4\sqrt{3}} \log \frac{x^2 + x\sqrt{3} + 1}{x^2 - x\sqrt{3} + 1} + \frac{1}{6} \arctan (2x + \sqrt{3})$$

$$+ \frac{1}{6} \arctan (2x - \sqrt{3}) + C.$$

2. Prove from Equation (3) that if $Q(x) = (x - a_1)(x - a_2) \dots (x - a_n)$, where $a_1 < a_2 < \dots < a_n$, then $\frac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \dots + \frac{r_n}{x - a_n} .$$

From Equation (3),

$$\frac{1}{Q(x)} = \frac{1}{a_1 - a_2}\left(\frac{1}{x - a_1} - \frac{1}{x - a_2}\right)\frac{1}{(x - a_3) \dots (x - a_n)} .$$

Thus $\frac{1}{Q(x)} = \left(\frac{1}{P_1(x)} - \frac{1}{P_2(x)}\right)\frac{1}{a_1 - a_2}$ where P_1 and P_2 are each products of $n - 1$ linear factors. Repeat the process, reducing the numbers of factors in each denominator by 1 at each step. After $n - 1$ steps, collect terms.

3. Prove if

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = b_n x^n + b_{n-1} x^{n-1} + \dots + b_0$$

for all but finitely many numbers x, that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0, 1, \dots, n$.

Consider $P(x) = \sum_{k=0}^{n} (a_k - b_k)x^k$. Since $P(x) = 0$ for more than n numbers x it follows since a polynomial of degree n can have at most n roots, that all the coefficients $a_k - b_k$ of P must vanish; otherwise a polynomial of degree less than or equal to n would have more than n roots.

4. Verify that $\int \frac{px + q}{[(x - a)^2 + b^2]} dx,$, $b > 0$, can be expressed as the sum of terms of the forms (11a, b, c).

As in the text, substitute $x = a + b \tan \theta$.

$$I = \frac{pa + q}{b} \arctan \frac{x - a}{b} + \frac{p}{2} \log [(x - a)^2 + b^2] + C.$$

A4-4. Definite Integrals

Solutions Exercises A4-4a

1. $\displaystyle\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + 99^2}\,dx.$

$I = 0$; the integrand is odd.

2. $\displaystyle\int_0^1 x^3 e^{-3x^2}\,dx$

Substitute $u = 3x^2$.

$$I = \frac{1}{18}\int_0^3 ue^{-u}\,du = -\frac{1}{18}(u + 1)e^{-u}\Big|_0^3 = \frac{1}{18}\left(1 - \frac{4}{e^3}\right).$$

3. $\displaystyle\int_1^e \log^3 x\,dx$

Integrate by parts with $dv = dx$, $u = \log^k x$, repeatedly, to obtain

$$I = 2(3 - e).$$

4. $\displaystyle\int_0^{\pi/2} \sin^m x\,dx$

From Exercises A4-2, Number 4(a),

$$I_m = \frac{m - 1}{m} I_{m-2}.$$

Since $I_0 = \frac{\pi}{2}$ and $I_1 = 1$, if m is even, $I = \frac{\pi}{2}\,\frac{1\cdot 3\cdot 5 \cdots m - 1}{2\cdot 4\cdot 6 \cdots m}$; if m is odd, $I = \frac{2\cdot 4\cdot 6 \cdots m - 1}{3\cdot 5\cdot 7 \cdots m}$.

5. $\displaystyle\int_0^{\pi/2} \sin^m x \cos^m x\,dx$, ($m$, a positive integer).

$$I = \int_0^{\pi/2} \frac{\sin^m 2x}{2^m}\,dx$$

$$= \frac{1}{2^{m+1}}\int_0^{\pi} \sin^m \theta\,d\theta$$

where $\theta = 2x$. As in Number 4, $I_m = \frac{m - 1}{m} I_{m-2}$; but here $I_0 = \pi$ and $I_1 = 2$.

6. $\displaystyle\int_0^{\pi/2} \frac{dx}{a + b \cos x}$, $a > b \geq 0.$

Set $x = 2 \arctan t$, $\cos x = \dfrac{1 - t^2}{1 + t^2}$, $dx = \dfrac{2\, dt}{1 + t^2}$.

$$I = \int_0^1 \frac{2}{(a + b) + (a - b)t^2}\, dt = \frac{2}{\sqrt{a^2 - b^2}} \arctan \sqrt{\frac{a - b}{a + b}} .$$

7. $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^3 x\, dx$

$$I = \int_0^{\pi/2} \sin^7 x\, (1 - \sin^2 x) \cos x\, dx = \frac{1}{8} - \frac{1}{10} = \frac{1}{40} .$$

8. $\displaystyle\int_1^2 \frac{dx}{x + x^5}$

$$I = \int_1^2 \frac{x^{-5}}{1 + x^{-4}}\, dx$$

Set $t = x^{-4}$. Then

$$I = \frac{1}{5} \int_{1/16}^1 \frac{dt}{1 + t} = \frac{1}{5} \log \frac{32}{17} .$$

9. $\displaystyle\int_0^b \sqrt{b^2 - x^2}\, dx = b^2 \int_0^{\pi/2} \cos^2 t\, dt = \frac{\pi}{4} b^2$ where $t = b \sin x$.

10. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta$, $a > 0$, $b > 0$.

$$I = \int_{-\pi/4}^{\pi/4} \frac{1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\, d\theta$$

since $\dfrac{\sin^5 \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ is odd. Set $t = \tan \theta$ to obtain

$$I = \int_{-1}^1 \frac{dt}{b^2 + a^2 t^2} = \frac{2}{ab} \arctan \frac{a}{b} .$$

11. Compare $\int_0^a f(x)dx$ with $\int_{-a}^0 f(x)dx$ when f is even or odd to derive the results (1) and (2) of the text by a method other than the one you employed for Exercises A4-2, Number 4.

Substitute $x = -t$ in $\int_0^a f(x)dx$ to obtain

$$\int_0^a f(x)dx = \int_{-a}^0 f(-x)dx.$$

Hence,

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

$$= \begin{cases} 0, & \text{if } f \text{ is odd,} \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even.} \end{cases}$$

12. Prove if f is integrable and periodic of period p, then for all a and b

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx.$$

Follow the geometrical approach of the text. Set $k = 1 + \left[\!\left[\frac{b-a}{p}\right]\!\right]$. Then $b < a + kp \leq b + p$. Consequently,

$$I = \int_b^{b+p} f(x)dx = \int_b^{a+kp} f(x)dx + \int_{a+kp}^{b+p} f(x)dx.$$

Now in the integral from b to $a + kp$ make the substitution $u = x - (k - 1)p$; in the integral from $a + kp$ to $b + p$, $u = x - kp$. Then

$$I = \int_{b-(k-1)p}^{a+p} f(u + kp)du + \int_a^{b-(k-1)p} f(u + (k - 1)p)du$$

$$= \int_{b-(k-1)p}^{a+p} f(u)du + \int_a^{b-(k-1)p} f(u)du$$

$$= \int_a^{a+p} f(u)du,$$

by Theorem A4-2b.

13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524.$$

If $0 \leq t \leq 1$, then $0 \leq t^n \leq t^2 \leq 1$ and $0 \leq 1 - t^2 \leq 1 - t^n \leq 1$. Thus

$$.500 = \int_0^{1/2} dt < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} \leq \int_0^{1/2} \frac{dt}{\sqrt{1 - t^2}}$$

$$\leq \arcsin \frac{1}{2} \leq \frac{\pi}{6} < .524\,.$$

14. Prove that $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\, dx = \pi^2$.

Since $\frac{x}{1 + \cos^2 x}$ is odd; $\frac{x \sin x}{1 + \cos^2 x}$, even,

$$I = 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = 4 \int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x}\, dx$$

$$= 2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx\,.$$

Hence,

$$I = -2\pi \arctan \cos x \Big|_0^{\pi} = -2\pi\left(- \frac{\pi}{4} - \frac{\pi}{4}\right) = \pi^2.$$

15. Show $\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1} \left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2$.

First, observe that $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n(n!)$; then that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n - 1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{2n!}{2^n(n!)}\,.$$

16. Determine the value exact to three deciaml places of

$$\int_1^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\, dx.$$

Set $\theta = \pi \log x$.

$$I = \frac{1}{\pi}\int_0^{36.1\pi} \sin\theta\, d\theta = -\frac{1}{\pi}\cos\theta\,\Big|_0^{\pi/10}$$

where $\int_0^{2\pi} \sin\theta\, d\theta = 0$ is used. Then

$$\cos\frac{\pi}{10} = 1 - \frac{\pi^2}{200} + \epsilon$$

where $0 < \epsilon \le \frac{1}{24}\left(\frac{\pi}{10}\right)^4$. Since $\pi^2 < 10$, the error term may be neglected to the desired accuracy. Hence, to the nearest thousandth,

$$I = \frac{\pi}{200} = .016.$$

17. Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\, dt.$

Observe that $\dfrac{t}{2 - \cos 2t}$ is odd; hence

$$I = \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{dt}{2 - \cos 2t}\,.$$

Set $u = \tan t$, $\cos 2t = \dfrac{1 - \tan^2 t}{1 + \tan^2 t} = \dfrac{1 - u^2}{1 + u^2}$, $dt = \dfrac{du}{1 + u^2}$. Then

$$I = \frac{\pi}{2}\int_0^1 \frac{du}{1 + 3u^2} = \frac{\pi}{2\sqrt{3}} \arctan u\sqrt{3}\,\Big|_0^1 = \frac{\pi^2}{6\sqrt{3}}\,.$$